THE FLOW OF AN INCOMPRESSIBLE
ELASTIC-PERFECTLY PLASTIC SOLID

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Flow of an Incompressible Elastic-Perfectly Plastic Solid", submitted by Harry Theodore Danyluk, B.Sc. (Honours), M.Sc. (Alberta), in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Two quasi-static, steady state flow problems of an incompressible, isotropic, elastic-perfectly plastic solid are considered in this thesis; namely, the plane radial flow through a rough converging channel and the axially symmetric radial flow through a rough converging conical channel. The classical PRANDTL-REUSS constitutive equations are rewritten in a form which is invariant under arbitrary changes of the frame of reference. This invariant form is necessary for the problems considered since the classical form is applicable only to problems which involve small deformations and for which the use of the infinitesimal strain tensor is justified.

The relationship between the form invariant PRANDTL-REUSS equations and the constitutive equations of a hypo-elastic solid of grade two is discussed. Also, two steady state flow problems of a hypo-elastic solid of grade two are considered: the plane radial flow through a smooth converging channel with and without consideration of inertia effects.

A frame indifferent definition of stress rate is of critical importance in modern theories of hypo-elasticity and elasto-plasticity. Consequently, a discussion on the definitions of stress rate is contained in this thesis.

An investigation is made of the nature of the governing equations for the plane strain flow of an elastic-perfectly plastic solid. The results of this investigation are applied in part to the plane radial flow problem of the elastic-perfectly plastic solid.

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NOMENCLATURE

α	semi-angle of converging channel (cone), 48(61)
α_i, i=1,2	functions of plastic strain rate invariants, 23
β	material constant μ/k, 67
Γ^i_{jk}	CHRISTOFFEL symbols of second kind, 40
γ	invariant, 44
γ	material constant $\sin^{-1} k/\mu$, 141
δ_{ij}	KRONECKER delta
$\frac{\delta\sigma_{ij}}{\delta t}$	GREEN-OLDROYD stress rate derivative, 34
$\frac{\delta T}{\delta t}$	convected time derivative of tensor T, 35
ε_{ij}	strain components
ε'_{ij}	strain deviation components
η	variable s_r/k , 67
λ	LAMÉ'S constant, 4
λ	non-negative scalar invariant, 10
λ_r	non-negative scalar invariants, 11
μ	modulus of rigidity or shear modulus, 4
ν	POISSON'S ratio, 84
ξ_i	LAGRANGIAN, convected or material coordinates, 29
ρ	density, 45
σ_{ij}	stress components
$\hat{\sigma}_{ij}$	COTTER-RIVLIN stress rate, 34
$\check{\sigma}_{ij}$	TRUESDELL stress rate, 34
τ	variable $s_{r\theta}/k$, 70

Φ	joint invariant $\sigma_{ij}\, d_{ji}$, 83
ϕ	anticlockwise rotation of an α-line from the x-axis, 92
Ψ	joint invariant $\sigma_{ij}\, \sigma_{jk}\, d_{ki}$, 83
ψ	angle between direction of algebraically greatest principal stress and a radial line, 52
ω_{ij}	vorticity components
$C_1, C_2, \ldots, C_{12}$	response coefficients, 81
d_{ij}	strain rate components
$d_{ij}^{(e)}$	elastic strain rate components, 17
$d_{ij}^{(P)}$	plastic strain rate components, 17
$\frac{D}{Dt}$	material derivative
$\frac{\mathcal{D}\sigma_{ij}}{\mathcal{D}t}$	JAUMANN or co-rotational stress rate, 34
F_i	co-variant body force components, 45
$f_{ij}^{(e)}$	elastic strain rate deviation components, 18
g_{mn}	components of metric tensor, 38
H_{ijkl}	components of hypo-elastic response function, 81
I_2	plastic strain rate invariant $d_{ij}^{(P)} d_{ij}^{(P)}$, 23
I_3	plastic strain rate invariant $d_{ij}^{(P)} d_{jk}^{(P)} d_{ki}^{(P)}$, 23
J_2'	second stress deviation invariant $1/2\ s_{ij}\ s_{ij}$, 7
J_3'	third stress deviation invariant $1/3\ s_{ij}\ s_{jk}\ s_{ki}$, 7
K	bulk modulus, 5
k	yield stress in pure shear, 7
n	material constant k/μ
n	degree of stress polynomial H_{ijkl}, 82
- p	hydrostatic part of stress tensor

P	variable p/2k, 97
s_{ij}	stress deviation components
Y	uniaxial yield stress, 7

CHAPTER I

CONSTITUTIVE EQUATIONS IN THE FLOW THEORY OF ELASTIC-PERFECTLY PLASTIC MEDIA FOR INFINITESIMAL STRAIN

This chapter contains a review of the classical elastic-perfectly plastic theory of PRANDTL-REUSS. This theory is applicable only to problems involving small deformations since approximations are made for the actual strains and stress rates. In this chapter all displacement components $u_i (i = 1,2,3)$ are referred to a fixed rectangular CARTESIAN frame $X_i (i = 1,2,3)$ and are assumed to be single-valued continuous functions of the spatial coordinates x_i or the material coordinates a_i and time t. The review begins with definitions characterizing small deformations and the usual CARTESIAN tensor notation is used throughout.

1.1 INFINITESIMAL STRAIN TENSORS AND STRAIN RATE TENSORS

The classical PRANDTL-REUSS theory is valid for small deformations, that is, for deformations that involve small displacements, small strains and small rotations. The precise meaning of the term small is discussed in TRUESDELL and TOUPIN [1].

GREEN'S strain tensor is given by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) + \frac{1}{2}\frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}$$

and if the terms $\tilde{E}_{i\ell}\,\tilde{E}_{j\ell}$, $\tilde{E}_{i\ell}\,\tilde{R}_{j\ell}$ and $\tilde{R}_{i\ell}\,\tilde{R}_{j\ell}$,

where

$$\tilde{E}_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right)$$

and

$$\tilde{R}_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial a_j} - \frac{\partial u_j}{\partial a_i}\right)$$

can be neglected in comparison with E_{ij}, then

$$E_{ij} = \tilde{E}_{ij} .$$

ALMANSI'S strain tensor is given by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{2} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}$$

and for small deformations

$$e_{ij} \cong \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) .$$

The displacements $u_i = x_i - a_i$ if small imply that

$$\frac{\partial u_i}{\partial x_j} \cong \frac{\partial u_i}{\partial a_j}$$

and

$$\frac{\partial x_i}{\partial a_j} = \delta_{ij} .$$

Consequently,

$$\varepsilon_{ij} \cong e_{ij} \cong E_{ij}$$

where

$$\varepsilon_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

defines the infinitesimal strain tensor and no distinction is made between spatial and material variables.

The material rate of change of GREEN'S strain tensor is given by

$$\frac{DE_{ij}}{Dt} = d_{st} \frac{\partial x_s}{\partial a_i} \frac{\partial x_t}{\partial a_j}$$

where

$$d_{st} \equiv \frac{1}{2} \left(\frac{\partial v_s}{\partial x_t} + \frac{\partial v_t}{\partial x_s}\right) \qquad (1.1.1)$$

are components of the rate of deformation tensor. The rate of deformation tensor is of importance in constitutive equations since it is a tensor that is independent of the rigid body rotation of a material. In plasticity literature d_{st} are usually called the components of the strain rate tensor and henceforth in this thesis this terminology will be used.

The infinitesimal strain increment during the infinitesimal time

increment dt may be defined as

$$d\eta_{ij} \equiv d_{ij}\, dt$$

and

$$d\eta_{ij} \equiv \frac{1}{2}\left[\frac{\partial(du_i)}{\partial x_j} + \frac{\partial(du_j)}{\partial x_i}\right]$$

where du_i is the displacement increment and x_i are spatial variables. Another definition of finite strain is

$$\eta_{ij} \equiv \int_0^t d_{ij}\, dt$$

where the integration is taken over the strain path.

1.2 ELASTIC STRESS-STRAIN RELATIONS FOR SMALL STRAINS

The stress-strain relation for the classical HOOKEAN solid is

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

where σ_{ij} are components of the stress tensor and λ and μ are the LAMÉ constants. This stress-strain relation can be expressed in terms of the stress deviation components s_{ij} and the strain deviation components ε'_{ij} as

$$s_{ij} = 2\mu\,\varepsilon'_{ij} \qquad (1.2.1a)$$

and

$$p = - K \varepsilon_{kk} , \qquad (1.2.1b)$$

where $p \equiv - \frac{1}{3} \sigma_{kk}$ and $K \equiv \lambda + \frac{2}{3} \mu$ is the bulk modulus. For an incompressible HOOKEAN solid,

$$\varepsilon_{kk} = 0 \text{ and } \varepsilon_{ij} = \varepsilon'_{ij} .$$

Equations (1.2.1a,b), differentiated with respect to time, gives

$$\dot{s}_{ij} = 2\mu\, f_{ij},\ f_{ij} \equiv d_{ij} - \frac{1}{3} d_{kk}\, \delta_{ij} ,$$

and (1.2.2a,b)

$$\dot{p} = - K\, d_{kk} .$$

The time derivative of stress is $\dot{s}_{ij} = \frac{\partial s_{ij}}{\partial t}$ since the deformation is small. For a HOOKEAN solid, the convected terms $v_k \frac{\partial s_{ij}}{\partial x_k}$ may be neglected, but for small deformation of an elastic-plastic solid these terms may not be neglected if the rate of work-hardening is small compared with E, YOUNG'S modulus of the material.

1.3 PERFECTLY PLASTIC STRESS-STRAIN RATE RELATIONS FOR SMALL DEFORMATIONS

Plasticity theory of the incremental or flow type relates the plastic strain increment or strain rate components and not the plastic

strain components with the current stress components. ST. VÉNANT [2] proposed that during two dimensional plastic deformation of a solid a co-axial relationship exists between the strain-rate tensor and the stress tensor. Independently, LÉVY [3] and VON MISES [4] proposed a similar relationship for three dimensional plastic deformation.

In the following derivations of the constitutive equations for the perfectly plastic theory of small deformations the approach used involves the concept of a yield function and its associated flow rule based on the existence of a potential function taken to be the yield function. This method is due to VON MISES [5] and MELAN [6].

According to the concept of a yield function there exists a function f of the stress such that

$$f(\sigma_{ij}) = C, \ C > 0, \tag{1.3.1}$$

and where C is a material parameter dependent on the strain history. Relationship (1.3.1) is called a yield criterion. The condition $f(\sigma_{ij}) < C$ identifies the elastic domain and $f(\sigma_{ij}) \geq C$ the plastic domain. For a non-hardening material, C is a material constant and $f(\sigma_{ij}) > C$ is an inadmissible state of stress. In this thesis, further discussion is confined strictly to non-hardening materials.

The conditions of isotropy and the independence of plastic deformation on hydrostatic pressure require f to be of the form

$$f(J_2', J_3') = C \tag{1.3.2}$$

where

$$J_2' \equiv \frac{1}{2} s_{ij} s_{ij}$$

and

$$J_3' \equiv \frac{1}{3} s_{ij} s_{jk} s_{ki}$$

are respectively the second and third stress deviation invariants. Absence of a BAUSCHINGER effect [$f(\sigma_{ij}) = f(-\sigma_{ij})$] requires f to be an even function in J_3' .

Two important yield criteria are the VON MISES yield criterion and the TRESCA yield criterion. The VON MISES yield criterion is

$$J_2' \equiv \frac{1}{2} s_{ij} s_{ij} = k^2 \tag{1.3.3}$$

which written in terms of the principal stress components σ_i $(i = 1,2,3)$ is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6k^2$$

where k is the yield stress in simple shear. The TRESCA yield criterion, although it can be expressed in the form of equation (1.3.2), is written more conveniently as

$$\sigma_{max} - \sigma_{min} = Y \tag{1.3.4}$$

where Y is the uniaxial yield stress and σ_{max} and σ_{min} are the maximum

and minimum principal stresses.

Depicted in a principal stress space with the principal stress σ_1, σ_2, and σ_3 taken as rectangular CARTESIAN coordinates, each criterion represents a surface called a yield surface. The VON MISES yield surface is a circular cylinder and the TRESCA yield surface is a regular hexagonal cylinder both infinite in extent and having their axes pass through the origin with direction numbers (1,1,1). Since plastic deformation is independent of the hydrostatic pressure, any section parallel to the plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ may be taken as representative of the stress domain. It is customary to take the orthogonal projection of a section parallel to σ_3 = constant onto the $\sigma_1 - \sigma_2$ plane. Figure 1 shows such a projection, the boundary of which is called a yield locus, and illustrates the relationship taken between the VON MISES yield locus and the TRESCA yield locus. The TRESCA yield locus is a linearization of the VON MISES yield locus.

Convexity of yield surfaces is fundamental in classical plasticity theory. A yield surface $f(\sigma_{ij}) = C$ is convex if

$$f(\sigma^*_{ij}) - f(\sigma_{ij}) \geq (\sigma^*_{ij} - \sigma_{ij}) \frac{\partial f}{\partial \sigma_{ij}} \qquad (1.3.5)$$

for any two arbitrary stress states σ^*_{ij} and σ_{ij} producing plastic deformation. Inequality in relationship (1.3.5) holds only for strictly convex surfaces. The convexity of the VON MISES and TRESCA yield surfaces are evident from a geometrical standpoint. However, convexity of any general yield surface cannot be shown using assumptions of perfect plasticity alone unless some additional hypothesis such as the principle of maximum work [7] is made.

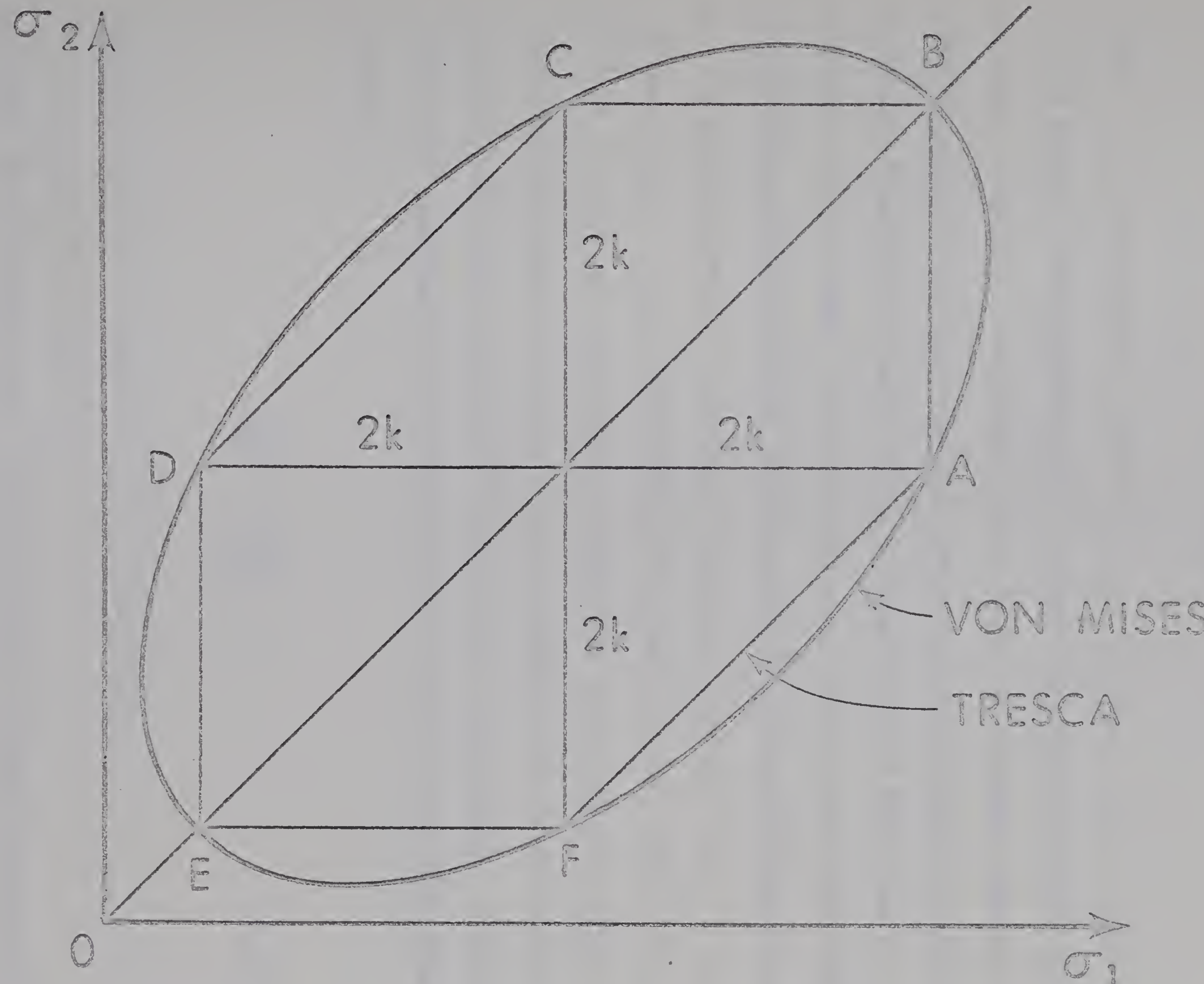


FIGURE 1 - THE VON MISES AND TRESCA YIELD LOCI

The hypothesis of the existence of a plastic potential states that there exists a function g of the stress such that the plastic strain rate components are given by the flow-rule

$$d_{ij}^{(P)} = \frac{\partial g}{\partial \sigma_{ij}} \lambda ,$$

where λ is a non-negative scalar invariant dependent only on coordinates x_i and time t. The dependence of g is upon the stress deviation invariants J_2' and J_3' in order that the co-axial relationship between the plastic strain-rate tensor and the stress deviation tensor be satisfied. A further hypothesis is made that g is identical to f [8]. Thus in principal stress space, the principal plastic strain rate vector $G\vec{d}(d_i^{(P)})$ associated with the principal stress vector $\vec{\sigma}(\sigma_i)$ has the same direction as the outward normal, if defined, to the yield surface $f(\sigma_{ij}) = C$ at the point $\sigma_i (i = 1,2,3)$ representing the plastic stress state. The $d_i^{(P)}$ $(i = 1,2,3)$ are the principal components of the plastic strain rate tensor and the factor G is used to give dimensions of stress to the plastic strain rate vector $\vec{d}(d_i^{(P)})$ in principal stress space. The plastic strain rate components for perfectly plastic materials are thus determined by

$$d_{ij}^{(P)} = \frac{\partial f}{\partial \sigma_{ij}} \lambda , \qquad (1.3.6)$$

where $\lambda = 0$ if $f < C$ and also if $f = C$ and $\dot{f} < 0$,

$\lambda \geq 0$ if $f = C$ and $\dot{f} = 0$.

An upper dot associated with a quantity denotes differentiation with respect to time or any other monotonically increasing parameter cor-

related with progressive deformation. The flow rule (1.3.6) corresponding to the particular yield function f is called the associated flow rule for f . The most general associated flow rule for f in the form given by equation (1.3.2) is

$$d_{ij}^{(P)} = \lambda[\frac{\partial f}{\partial J_2'} s_{ij} + \frac{\partial f}{\partial J_3'} (s_{ik} s_{jk} - \frac{2}{3} J_2' \delta_{ij})] .$$

Components of the plastic strain rate tensor are determined up to an arbitrary non-negative scalar factor at a plastic stress state provided $\frac{\partial f}{\partial \sigma_{ij}}$ are uniquely defined there. This is possible for all points on the yield locus if the corresponding yield surface is a regular surface. The VON MISES yield surface is regular; hence the plastic strain rate components are derivable using the flow rule (1.3.6) to give

$$d_{ij}^{(P)} = \lambda s_{ij} , \qquad (1.3.7)$$

the LÉVY-MISES flow relation. If, however, the yield surface is a singular surface, the corresponding associated flow rule is given by the KOITER generalization of the plastic potential [9]. According to this generalization, the plastic strain rate components are determined by

$$d_{ij}^{(P)} = \sum_r \frac{\partial f_r}{\partial \sigma_{ij}} \lambda_r . \qquad (1.3.8)$$

λ_r are non-negative scalar invariants dependent only on coordinates x_i and time t and

$$\lambda_r = 0 \text{ if } f_r < C \text{ and also if } f_r = C, \dot{f}_r < 0 ,$$

$$\lambda_r \geq 0 \text{ if } f_r = C \text{ and } \dot{f}_r = 0.$$

The summation is taken over all surfaces $f_r = C$ whose join forms the yield surface in the neighbourhood of the plastic stress point considered. In principal stress space, the direction of $G\vec{d}(d_i^{(P)})$ at a singular point must be within the region spanned by the unique normals drawn outwards to the faces intersecting at the particular singular point considered.

There is only one plastic regime available as the stress point traverses the VON MISES yield locus. However the TRESCA yield surface is singular and as the stress point traverses the TRESCA yield locus different plastic regimes are available. If the stress point lies on a flat of the yield locus then, from the concept of the plastic potential,

$$d_{max}^{(P)} = \lambda = - d_{min}^{(P)} , \; d_{int}^{(P)} = 0 ,$$

where $d_{max}^{(P)}$, $d_{int}^{(P)}$ and $d_{min}^{(P)}$ are the maximum, intermediate and minimum principal components of the plastic strain rate and $\lambda \geq 0$. If the stress point lies on a corner, for example point B of Figure 1, it follows from KOITER'S generalization of the plastic potential that

$$d_1^{(P)} = \lambda_1, \; d_2^{(P)} = \lambda_2, \; d_3^{(P)} = - (\lambda_1 + \lambda_2)$$

where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$. For point C

$$d_1^{(P)} = - \lambda_1, \; d_2^{(P)} = \lambda_1 + \lambda_2, \; d_3^{(P)} = - \lambda_2$$

where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

The form of the scalar invariant λ in the LÉVY-MISES flow relation (1.3.7) is determined by squaring this equation to give

$$d_{ij}^{(P)}\ d_{ij}^{(P)} = \lambda^2\ s_{ij}\ s_{ij} = 2\lambda^2 J_2' \ .$$

Since $\frac{1}{2}\, d_{ij}^{(P)}\ d_{ij}^{(P)} \equiv I_2'$, the second invariant of $d_{ij}^{(P)}$, it follows that

$$\lambda = \sqrt{\frac{I_2'}{J_2'}} = \frac{\sqrt{I_2'}}{k} = \frac{\sqrt{d_{ij}^{(P)}\ d_{ij}^{(P)}}}{2\ k} \ .$$

λ is thus a function of the invariants of the plastic strain rate tensor and stress deviation tensor. Alternatively, multiplying both sides of equation (1.3.7) by s_{ij} gives

$$d_{ij}^{(P)}\ s_{ij} = \lambda\ s_{ij}\ s_{ij} = 2\lambda J_2'$$

from which

$$\lambda = \frac{d_{ij}^{(P)}\ s_{ij}}{s_{\ell m}\ s_{\ell m}} = \frac{d_{ij}^{(P)}\ s_{ij}}{2k^2} \ .$$

The LÉVY-MISES flow relation for perfectly plastic theory is now rewritten as

$$d_{ij}^{(P)} = \frac{d_{\ell m}^{(P)}\ s_{\ell m}}{2k^2}\ s_{ij} \qquad (1.3.9a)$$

or

$$d_{ij}^{(P)} = \frac{\sqrt{d_{\ell m}^{(P)}\, d_{\ell m}^{(P)}}}{\sqrt{2}\, k}\, s_{ij} \,. \qquad (1.3.9b)$$

Each relation (1.3.9a or b) is homogeneous of order one in the plastic strain rates. Thus perfectly plastic flow is inviscid.

1.4 ST. VÉNANT-LÉVY-MISES ELASTIC-PERFECTLY PLASTIC THEORY

The classical theory describing the response of an elastic-perfectly plastic material in which no account is made for elastic effects during plastic deformation is called the ST. VÉNANT-LÉVY-MISES theory. If the yielding of the material is governed by the VON MISES yield criterion, the theory is referred to as the LÉVY-MISES theory [10].

For analysis employing the ST. VÉNANT-LÉVY-MISES theory, two sets of constitutive equations are required; one for the region deforming elastically and the other for the region deforming plastically. For the elastic region, the constitutive equations are given by

$$s_{ij} = 2\mu\, \varepsilon_{ij} \,, \; \varepsilon_{kk} = 0 \,,$$

for incompressible deformations and by

$$s_{ij} = 2\mu\, \varepsilon'_{ij} \,, \; p = -\, K\varepsilon_{kk} \,,$$

for compressible deformations. For the plastic region, the constitutive equations depend upon the yield criterion used. With the VON MISES yield criterion (1.3.3) written as

$$f \equiv s_{ij}\, s_{ij} = 2k^2 , \tag{1.4.1}$$

the constitutive equations are the LÉVY-MISES relation (1.3.7), namely,

$$d_{ij}^{(P)} = \lambda\, s_{ij} \tag{1.4.2}$$

where

$$\lambda \geq 0 \text{ for } f = 2k^2 \text{ and } \dot{f} = 0.$$

With the TRESCA yield criterion (1.3.4), the constitutive equations are

$$d_{ij}^{(P)} = \sum_r \frac{\partial f_r}{\partial \sigma_{ij}} \lambda_r \tag{1.4.3}$$

where

$$\lambda_r \geq 0 \text{ for } f_r = 2k \text{ and } \dot{f}_r = 0.$$

In any specific elastic-perfectly plastic deformation problem where the equations (1.4.3) are used, the plastic regime applicable must first be determined. However this is not always possible and an additional assumption is made to make the problem either statically or kinematically determinate. This amounts to specifying the singular point or flat used on the TRESCA yield locus. One such assumption is the HAAR-KÁRMÁN hypothesis [11] which states that in some axially symmetric perfectly plastic problems, the circumferential stress is equal to one of the principal

stresses in the meridional plane. This situation does not occur for the LÉVY-MISES theory since there is only one plastic regime available. However, the LÉVY-MISES theory has one undesirable property in that the governing equations for axially symmetric flow problems are non-hyperbolic in character thus adding difficulties to the mathematical analysis [12].

A feature of the ST. VÉNANT-LÉVY-MISES theory which causes difficulty is the determination of the elastic-plastic boundary.

If in the deformation of an elastic-perfectly plastic material, elastic deformations are taken as vanishing identically so that the deformation of the material is just that due to the perfectly plastic deformation, one is then led to the 'rigid-plastic' theory. This theory is applicable strictly to a hypothetical 'rigid-plastic' material which remains rigid for stress states below yield and whose elastic moduli are indefinitely large. The constitutive equations are given by equation (1.4.2) or equation (1.4.3) with $d_{ij}^{(P)}$ replaced by d_{ij}. This theory has been applied to the problems involving unconstrained plastic flow of an elastic-perfectly plastic solid where the elastic deformations are small compared to the plastic deformations and so may be justifiably neglected provided the analysis is not in the immediate vicinity of the transition zone between elastic and plastic zones.

1.5 CLASSICAL ELASTIC-PERFECTLY PLASTIC THEORY OF PRANDTL-REUSS

The incorporation of the elastic strains in the stress-rate of strain relations for a perfectly plastic solid was proposed by PRANDTL [13] for two dimensional flow problems and by REUSS [14] for three dimensional flow problems. The definitions of strain and stress rate used are applicable only for small deformations.

During the continued loading of an element of an elastic-perfectly plastic material deforming plastically, the total displacement of each material point of the element may be represented by

$$u_i = u_i^{(e)} + u_i^{(P)}$$

where $u_i^{(e)}$ is the displacement vector corresponding to the elastic deformation, and $u_i^{(P)}$ is the displacement vector corresponding to the plastic deformation remaining after unloading. The decomposition of u_i into elastic and plastic components is not unique since $u_i^{(e)}$ and $u_i^{(P)}$ are determined up to a rigid body displacement of the whole material. The same remarks apply to the velocity vector given by

$$v_i = v_i^{(e)} + v_i^{(P)} \tag{1.5.1}$$

where $v_i^{(e)}$ and $v_i^{(P)}$ are the velocity vectors corresponding to the elastic and plastic deformations respectively. Consequently, the strain rate tensor defined by equation (1.1.1) is used in formulating the classical PRANDTL-REUSS equations since it is a tensor independent of the rigid body rotation of the material. From equation (1.5.1) it follows that

$$d_{ij} = d_{ij}^{(e)} + d_{ij}^{(P)} \tag{1.5.2}$$

where $d_{ij}^{(e)}$ and $d_{ij}^{(P)}$ are respectively the uniquely determined components of the elastic and plastic strain rate tensors defined by

$$d_{ij}^{(e)} \equiv \frac{1}{2} \left(\frac{\partial v_i^{(e)}}{\partial x_j} + \frac{\partial v_j^{(e)}}{\partial x_i} \right)$$

and

$$d_{ij}^{(P)} \equiv \frac{1}{2}\left(\frac{\partial v_i^{(P)}}{\partial x_j} + \frac{\partial v_j^{(P)}}{\partial x_i}\right).$$

In further discussion a solid which yields plastically according to the VON MISES yield criterion will be called a MISES solid and one which yields plastically according to the TRESCA yield criterion will be called a TRESCA solid.

For an elastic-perfectly plastic MISES solid which is elastically compressible and which is deforming plastically, the strain rate tensor is obtained from equation (1.5.2) rewritten as

$$d_{ij} = f_{ij}^{(e)} + \frac{1}{3} d_{kk}^{(e)} \delta_{ij} + d_{ij}^{(P)} \tag{1.5.3}$$

where $f_{ij}^{(e)}$ are components of the elastic strain rate deviation tensor. From equations (1.2.2a,b) it then follows that

$$d_{ij} = \frac{1}{2\mu} \frac{Ds_{ij}}{Dt} - \frac{1}{3K} \frac{Dp}{Dt} \delta_{ij} + \lambda s_{ij} . \tag{1.5.4a}$$

This together with

$$d_{kk}^{(e)} = d_{kk} = - \frac{1}{K} \frac{Dp}{Dt} \tag{1.5.4b}$$

constitute the constitutive equations for the MISES solid. The operator $\frac{D}{Dt}$ is the material derivative defined by $\frac{D}{Dt}(\ \ ) \equiv \frac{\partial}{\partial t}(\ \ ) + v_k \frac{\partial}{\partial x_k}(\ \ )$. However, differentiation need not be with respect to time t but to any other monotonically increasing parameter correlated with progressive de-

formation. This is possible since plastic deformation is inviscid.

If $d_{kk}^{(e)} = 0$, it follows from equations (1.5.4a,b) that

$$d_{ij} = \frac{1}{2\mu} \frac{Ds_{ij}}{Dt} + \lambda s_{ij}$$

and (1.5.5a,b)

$$d_{kk} = 0.$$

Equations (1.5.5a,b) are the constitutive equations for an incompressible elastic-perfectly plastic MISES solid.

An expression for λ in the case $\lambda \neq 0$ can be obtained by multiplying each side of equation (1.5.4a) or equation (1.5.5a) by s_{ij} and using the result on differentiating materially the VON MISES yield criterion (1.4.1) that

$$\frac{Df}{Dt} = 2s_{ij} \frac{Ds_{ij}}{Dt} = 0.$$

This, along with the VON MISES yield criterion, yields

$$\lambda = \frac{d_{ij} s_{ij}}{2k^2} . \qquad (1.5.6)$$

For an elastic-perfectly plastic TRESCA solid which is elastically compressible and deforming plastically, the constitutive equations are

$$d_{ij} = \frac{1}{2\mu}\frac{Ds_{ij}}{Dt} - \frac{1}{3K}\frac{Dp}{Dt}\delta_{ij} + \sum_r \lambda_r \frac{\partial f_r}{\partial s_{ij}}$$

and (1.5.7a,b)

$$d_{kk} = -\frac{1}{K}\frac{Dp}{Dt}\,.$$

The role of λ_r is as given in relationship (1.3.8). Equations (1.5.7a,b) follow analogously from equations (1.5.3) as did equations (1.5.4a,b) on simply replacing $d_{ij}^{(P)}$ by its form given in equation (1.3.8).

If $d_{kk} = d_{kk}^{(e)} = 0$, then from equations (1.5.7a,b) one obtains

$$d_{ij} = \frac{1}{2\mu}\frac{Ds_{ij}}{Dt} + \sum_r \lambda_r \frac{\partial f_r}{\partial s_{ij}}$$

and (1.5.8a,b)

$$d_{kk} = 0.$$

These are the constitutive equations for an incompressible elastic-perfectly plastic TRESCA solid. The role of λ_r is as given in relation (1.3.8).

From the constitutive equations of the classical PRANDTL-REUSS equations considered in the foregoing discussion, those for the rigid plastic theory are derivable on letting μ be indefinitely large, provided that all stress derivatives and velocities are finite.

On first neglecting convected terms of type $v_k \dfrac{\partial s_{ij}}{\partial x_k}$, inte-

gration with respect to time t of equations (1.5.4a,b) or (1.5.7a,b) with λ, λ_r = 0 yields

$$\varepsilon'_{ij} = \frac{1}{2\mu} s_{ij}$$

and

$$\varepsilon_{kk} = - \frac{p}{K} .$$

These are the constitutive equations of a compressible HOOKEAN solid for infinitesmal strains under the usual initial conditions of a stress free-strain free natural state. Similarly from equations (1.5.5a,b) or (1.5.8a,b), one obtains

$$\varepsilon_{ij} = \frac{1}{2\mu} s_{ij}$$

and

$$\varepsilon_{kk} = 0 ,$$

the constitutive equations of an incompressible HOOKEAN solid.

The constitutive equations of the classical PRANDTL-REUSS theory are similar to those of ST. VÉNANT-LÉVY-MISES theory with the main exception that elastic deformations are accounted for in determining the deformations occurring during plastic flow. There are still two sets of constitutive equations needed; one valid for the elastic region and the other in the plastic region. For loading systems passing from elastic to plastic

states, a determination of the elastic-plastic boundary is still required. However, the constitutive equations have been made more complicated by the introduction of stress rates.

Attempts at formulating an elastic-perfectly plastic theory which would lead to a gradual transition from elastic to plastic states with the use of just a single set of constitutive equations valid for any stress state not violating the yield criterion, were made by PRAGER [15] and THOMAS [16]. THOMAS'S work is a generalization and extension of an earlier work [17] on axiomatizing a perfectly plastic theory valid for infinitesimal strain. This latter work is presently considered.

1.6 PERFECTLY PLASTIC THEORY OF T.Y. THOMAS

T.Y. THOMAS [17] proposed a theory aimed at revealing the possible forms that the constitutive equations of a perfectly plastic material may take on using established characteristics of this material under plastic deformation. To formulate this new mathematical science, THOMAS proposed:

Axiom 1. Perfectly plastic deformation occurs without volume change, that is,

$$d_{kk}^{(P)} = 0 \; ;$$

Axiom 2. The stress deviation tensor is an isotropic tensor function of the plastic strain rate deviation tensor, that is,

$$s_{ij} = \Phi_{ij} \, (f_m^{(p)}) \; ; \qquad (1.6.1)$$

Axiom 3. Relationship (1.6.1) does not establish a one-to-one correspondence between the stress deviation tensor and the plastic strain rate deviation tensor.

With these axioms, THOMAS obtained the constitutive equations for the LÉVY-MISES theory and constitutive equations for a perfectly plastic TRESCA solid. This latter set of constitutive equations differ from those given previously in SECTION 1.3. The perfectly plastic theory proposed by THOMAS is not consistent with the perfectly plastic theory based on the concept of a plastic potential identified as the yield function.

Axioms 1 and 2 imply that

$$s_{ij} = \Phi_{ij}(I_2, I_3) \tag{1.6.2}$$

where $I_2 \equiv d_{ij}^{(P)} \; d_{ij}^{(P)}$ and $I_3 \equiv d_{ij}^{(P)} \; d_{jk}^{(P)} \; d_{ki}^{(P)}$ are the two non-zero basic invariants of the plastic strain rate tensor. Since Φ_{ij} are components of a symmetric second order tensor and dependent on invariants I_2 and I_3 and since $s_{ii} = 0$, equation (1.6.2) can be expressed [18] as

$$s_{ij} = \alpha_1 \, d_{ij}^{(P)} + \alpha_2 \, (d_{ik}^{(P)} \; d_{kj}^{(P)} - \frac{1}{3} I_2 \, \delta_{ij}) \tag{1.6.3}$$

where $\alpha_i = \alpha_i \, (I_2, I_3)$, $i = 1,2$. Multiplication of equation (1.6.3) by $\frac{1}{2} s_{ij}$ and $\frac{1}{3} s_{jk} \, s_{ki}$ alternately yields

$$J_2' = \frac{\alpha_1^{\,2}}{2} I_2 + \alpha_1 \alpha_2 I_3 + \frac{1}{12} \alpha_2^{\,2} I_2^{\,2} \tag{1.6.4.a}$$

and

$$J_3' = \frac{\alpha_1^{\ 3}}{3} I_3 + \frac{1}{6}\alpha_1^{\ 2}\alpha_2 I_2^{\ 2} + \frac{1}{6}\alpha_1\alpha_2^{\ 2} I_2 I_3 + \frac{1}{9}\alpha_2^{\ 3}(I_3^{\ 2} - \frac{1}{12} I_2^{\ 3}). \tag{1.6.4b}$$

By axiom 3, the JACOBIAN Δ of the equations (1.6.4a,b) is zero. This results in a partial differential equation involving the unknown α_1 and α_2. Guided by the existing forms of the previously established constitutive equations of the LÉVY-MISES theory, THOMAS considered the specific case where $\alpha_2 = 0$ and $\alpha_1 > 0$. The quasi-linear partial differential equation

$$2I_2 \frac{\partial\alpha_1}{\partial I_2} + 3I_3 \frac{\partial\alpha_1}{\partial I_3} + \alpha_1 = 0$$

so formed has the general solution

$$\alpha_1 = \frac{1}{\sqrt{I_2}} F(\frac{\sqrt[3]{I_3}}{\sqrt{I_2}}) = \frac{1}{\sqrt{I_2}} F(\frac{\sqrt[3]{3J_3'}}{\sqrt{2J_2'}})$$

for arbitrary differentiable function F. Thus a general form for the constitutive equations of perfect plasticity is

$$s_{ij} = \frac{1}{\sqrt{I_2}} F(\frac{\sqrt[3]{3J_3'}}{\sqrt{2J_2'}})\, d_{ij}^{(P)}. \tag{1.6.5}$$

Squaring equation (1.6.5), one obtains the relation

$$s_{ij}\, s_{ij} - [F\,(\frac{\sqrt[3]{3J_3'}}{\sqrt{2J_2'}})]^2 = 0 \tag{1.6.6}$$

which plays the role of a yield criterion since it is a restriction on the stress deviation invariants. For F a constant function defined by

$$F(J_2', J_3') = \sqrt{2}k, \text{ k a material constant,}$$

equation (1.6.6) becomes

$$s_{ij}\, s_{ij} = 2k^2 ,$$

the VON MISES yield criterion with the appropriate interpretation of k. Also equation (1.6.5) becomes

$$d_{ij}^{(P)} = \frac{\sqrt{d_{\ell m}^{(P)}\, d_{\ell m}^{(P)}}}{\sqrt{2}k}\, s_{ij}$$

$$= \frac{d_{\ell m}^{(P)}\, s_{\ell m}}{2k^2}\, s_{ij} ,$$

the constitutive equations for the LÉVY-MISES theory.

Alternative representations of equation (1.6.5) and equation (1.6.6) are available if equation (1.6.5) is expressed in terms of s_i and $d_i^{(P)}$, the principal components of the stress deviation tensor and plastic strain rate tensor respectively. If $d_i^{(P)}$ are ordered such that $d_1^{(P)} > d_2^{(P)} > d_3^{(P)}$, then equation (1.6.5) can be written as

$$s_i = \frac{G(d_3^{(P)}/d_1^{(P)})}{d_1^{(P)} - d_3^{(P)}}\, d_i^{(P)} \tag{1.6.7}$$

for arbitrary differentiable function G > 0. From this one gets

$$s_1 - s_3 = G(\frac{s_3}{s_1}) , \tag{1.6.8}$$

a relation analogous to relation (1.6.6). Choosing $G(\frac{s_3}{s_1}) = Y$, Y a material constant, equation (1.6.8) becomes

$$s_1 - s_3 = Y ,$$

the TRESCA yield criterion for plastic regime AB in Figure 1 for Y interpreted as the yield stress in simple shear. The corresponding constitutive equations are

$$s_i = \frac{Y}{d_1^{(P)} - d_3^{(P)}} d_i^{(P)} \tag{1.6.9}$$

as follows from equation (1.6.7). These, however, differ from the constitutive equations for plastic regime AB using the concept of the plastic potential since then $d_2^{(P)} = 0$, whereas from equation (1.6.9), $d_2^{(P)} \neq 0$ necessarily.

CHAPTER II

CONSTITUTIVE EQUATIONS OF THE ELASTIC-PERFECTLY PLASTIC THEORY OF PRANDTL-REUSS FOR FINITE STRAIN

The constitutive equations for the finite elastic-perfectly plastic theory of PRANDTL-REUSS are developed in this chapter. This necessarily involves a preliminary discussion on invariant forms of constitutive equations; a topic of recent interest in non-linear field theory of mechanics. The development of this topic has been varied both in approach and in notation.

As seen in Chapter I, the constitutive equations of the classical PRANDTL-REUSS theory are suitable only for problems involving infinitesimal strains and rotations. A generalization of these equations valid with respect to a fixed reference system and applicable for finite strains has been achieved by THOMAS [18]. Primarily it is the presence of the material derivative of the stress tensor that invalidates the use of the classical PRANDTL-REUSS equations in problems involving finite deformation. The reason for this is that these constitutive equations violate the 'principle of material objectivity', the invariance principle of constitutive equations under arbitrary changes of the frame of reference [19].

The local motion of a deforming material element surrounding a point P can be resolved into a translation, an instantaneous rigid body rotation about an axis through P, a spherical dilatation and a pure shear deviation. The components of this local motion involving rotation and translation do not produce deformation of the element, hence constitutive equations can best be formulated with respect to a rectangular

CARTESIAN coordinate reference system at P moving such that the local rotation and translation components vanish. Such a coordinate reference system has been termed a 'kinematically preferred coordinate system' by THOMAS [20] and a 'co-rotational reference system' in TRUESDELL and TOUPIN [21]. A more general approach was employed by OLDROYD [22] whereby the constitutive equations for homogeneous materials are formulated with respect to a 'convected' coordinate system which moves and deforms with the material. NOLL [23] in his earlier work on invariant forms of constitutive equations uses the term 'principle of isotropy of space' and RIVLIN and ERICKSEN [24] uses the term 'form invariance under rotation of the physical system'. Since physical properties of material are independent of the coordinate reference system used, the object of the above works was to determine, within the framework of classical mechanics, quantities associated with the material which are independent of the rigid body motions of this material. More generally, OLDROYD sought invariant forms which were independent of any arbitrary motion of the material. It is to be observed that the less restrictive 'principle of material objectivity' mentioned earlier requires that the constitutive equations be invariant under any arbitrary change of the frame of reference and hence of the observer.

In SECTION 2.1 of this chapter, physical quantities are determined which are invariant under arbitrary rigid body motions considered as elements of the proper orthogonal transformation group. Such quantities are called objective and attention will be confined only to those which are needed in the formulation of the constitutive equations of the PRANDTL-REUSS theory for finite strain. Whether or not the rigid body motions are subjected to the material body and the applied

forces or to the coordinate reference system is of no consequence under this approach. This would not be true, however, if the rigid body motions were considered as elements of the full orthogonal transformation group. Reflection would then be a permissible operation but this is physically meaningless if applied to material bodies. All coordinate reference systems used are rectangular CARTESIAN systems and time is the absolute time of classical mechanics. Thus the approach taken here is the same as that considered in THOMAS [20] and ERINGEN [25] but differs from that in GREEN and ADKINS [26] and NOLL [27].

2.1 OBJECTIVE TENSORS AND STRESS RATES

Consider a deforming material body moving through space and let x_i and $\overline{x}_i (i = 1,2,3)$ be the spatial coordinates of the same material point with respect to two coordinate reference frames S and $\overline{S}$. Then the two motions $x_i\ (\xi_1, \xi_2, \xi_3, t) \equiv x_i\ (\xi_i, t)$ and $\overline{x}_i\ (\xi_1, \xi_2, \xi_3, \tau) \equiv \overline{x}_i\ (\xi_i, \tau)$ are said to be objectively equivalent if and only if

$$\overline{x}_i\ (\xi_i, \tau) = a_{ij}(t)\ x_j\ (\xi_i, t) + b_i(t)$$

where (2.1.1a,b)

$$a_{ij}(t)\ a_{ik}(t) = a_{ji}(t)\ a_{ki}(t) = \delta_{jk}\ , \ |a_{ij}(t)| = +\ 1$$

and $\tau = t - a$, a being a constant. The absolute times t and τ used with the S and $\overline{S}$ systems respectively can have different fiducial time origins. The $\xi_i (i = 1,2,3)$ are the LAGRANGIAN, convected or material coordinates of the material particle and may be taken as the coordinates

of the particle in either S or $\overline{S}$ at some specific instant of time provided a is known. Any tensor associated with the material particle is also said to be objective if in any two objectively equivalent motions it obeys its appropriate tensor transformation law for all times.

From equation (2.1.1a), one obtains

$$\overline{v}_i(\tau) \equiv \frac{D\overline{x}_i(\tau)}{D\tau} = a_{ij}(t)\ v_j(t) + x_j(t)\ \frac{Da_{ij}(t)}{Dt} + \frac{Db_i(t)}{Dt} \tag{2.1.2}$$

where $\frac{D}{D\tau}$ and $\frac{D}{Dt}$ denote differentiation with respect to time keeping the convected coordinates ξ_i constant. Equation (2.1.2) shows that velocity is not objective and it follows likewise using equation (2.1.2) that acceleration is not objective. Differentiation of equation (2.1.2) with respect to $\overline{x}_j(\tau)$ gives

$$\overline{v}_{i,j}(\tau) = a_{ik}(t)\ v_{k,m}(t)\ \frac{\partial x_m(t)}{\partial \overline{x}_j(\tau)} + \frac{Da_{ik}(t)}{Dt}\ \frac{\partial x_k(t)}{\partial \overline{x}_j(\tau)}\ . \tag{2.1.3}$$

Since

$$x_k(t) = a_{jk}(t)\ \overline{x}_j(\tau) - b_j(t)\ a_{jk}(t),$$

equation (2.1.3) is written as

$$\overline{v}_{i,j}(\tau) = a_{ik}(t)\ a_{jm}(t)\ v_{k,m}(t) + a_{jk}(t)\ \frac{Da_{ik}(t)}{Dt} \tag{2.1.4}$$

from which

$$\bar{v}_{j,i}(\tau) = a_{ik}(t)\, a_{jm}(t)\, v_{m,k}(t) + a_{ik}(t) \frac{Da_{jk}(t)}{Dt} . \quad (2.1.5)$$

Moreover, from equation (2.1.1b),

$$a_{ik}(t) \frac{Da_{jk}(t)}{Dt} = - a_{jk} \frac{Da_{ik}(t)}{Dt} \quad (2.1.7)$$

and on defining

$$\alpha_{ij}(t) \equiv a_{ik}(t) \frac{Da_{jk}(t)}{Dt} ,$$

equations (2.1.4) and (2.1.5) are written as

$$\bar{v}_{i,j}(\tau) = a_{ik}(t)\, a_{jm}(t)\, v_{k,m}(t) - \alpha_{ij}(t)$$

and (2.1.7a,b)

$$\bar{v}_{j,i}(\tau) = a_{ik}(t)\, a_{jm}(t)\, v_{m,k}(t) + \alpha_{ij}(t)$$

respectively. From equations (2.1.7a,b), one then obtains the ZORAWSKI relations [28]

$$\bar{d}_{ij}(\tau) = a_{ik}(t)\, a_{jm}(t)\, d_{km}(t)$$

and (2.1.8a,b)

$$\bar{\omega}_{ij}(\tau) = a_{ik}(t)\, a_{jm}(t)\, \omega_{km}(t) - \alpha_{ij}(t) ,$$

where ω_{ij} are components of the vorticity tensor defined by

$$\omega_{ij} \equiv \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) . \qquad (2.1.9)$$

Thus the strain rate tensor is objective while the vorticity tensor is not.

With respect to the S system, the stress components at a material point P of the stressed medium at time t are denoted by $\sigma_{ij}(t)$ and with respect to the $\overline{S}$ system they are denoted by $\overline{\sigma}_{ij}(\tau)$, $\tau = t - a$. The spatial coordinate dependence of the stress tensors is understood in the notation. Due to the objective tensor character of stress,

$$\overline{\sigma}_{ij}(\tau) = a_{im}(t)\, a_{jn}(t)\, \sigma_{mn}(t) . \qquad (2.1.10)$$

Taking the material derivative of both sides of equation (2.1.10), one obtains

$$\frac{D\overline{\sigma}_{ij}(\tau)}{D\tau} = a_{im}(t)\, a_{jn}(t) \frac{D\sigma_{mn}(t)}{Dt} + a_{im}(t) \frac{Da_{jn}(t)}{Dt} \sigma_{mn}(t)$$

$$+ a_{jn}(t) \frac{Da_{im}(t)}{Dt} \sigma_{mn}(t) \qquad (2.1.11)$$

which shows that the material derivative of the stress tensor is not objective. However, from equation (2.1.11), objective quantities can be obtained. For from equations (2.1.4) and (2.1.5),

$$\frac{Da_{jn}(t)}{Dt} = a_{in}(t)\, \overline{v}_{j,i}(\tau) - a_{ji}(t)\, v_{i,n}(t) , \qquad (2.1.12a)$$

$$= - a_{in}(t)\, \bar{v}_{ij}(\tau) + a_{ji}(t)\, v_{n,i}(t). \qquad (2.1.12b)$$

Addition of equations (2.1.12a,b) and use of equation (2.1.9) yields

$$\frac{Da_{jn}(t)}{Dt} = a_{in}(t)\, \bar{\omega}_{j,i}(\tau) - a_{ji}(t)\, \omega_{in}(t). \qquad (2.1.13)$$

Substitution of equations (2.1.12a), (2.1.12b) and (2.1.13) alternately into equation (2.1.11) gives

$$\frac{\delta}{\delta t}\, \bar{\sigma}_{ij}(\tau) = a_{ir}(t)\, a_{js}(t)\, \frac{\delta}{\delta t}\, \sigma_{rs}(t)\ ,$$

$$\hat{\bar{\sigma}}_{ij}(\tau) = a_{ir}(t)\, a_{js}(t)\, \hat{\sigma}_{rs}(t)\ , \qquad (2.1.14a,b,c)$$

and

$$\frac{\mathcal{D}\bar{\sigma}_{ij}(\tau)}{\mathcal{D}t} = a_{ir}(t)\, a_{js}(t)\, \frac{\mathcal{D}\sigma_{rs}(t)}{\mathcal{D}t},$$

where

$$\frac{\delta}{\delta t}\, \sigma_{ij}(t) \equiv \frac{D\sigma_{ij}(t)}{Dt} - v_{j,m}(t)\, \sigma_{im}(t) - v_{i,m}(t)\, \sigma_{mj}(t),$$

$$\hat{\sigma}_{ij}(t) \equiv \frac{D\sigma_{ij}(t)}{Dt} + v_{m,j}(t)\, \sigma_{im}(t) + v_{m,i}(t)\, \sigma_{mj}(t), \qquad (2.1.15a,b,c)$$

and

$$\frac{\mathcal{D}\sigma_{ij}(t)}{\mathcal{D}t} = \frac{D\sigma_{ij}(t)}{Dt} + \sigma_{im}(t)\,\omega_{mj}(t) + \sigma_{jm}(t)\,\omega_{mi}(t)\ .$$

Thus $\frac{\delta\sigma_{ij}}{\delta t}$, $\hat{\sigma}_{ij}$ and $\frac{\mathcal{D}\sigma_{ij}}{\mathcal{D}t}$ are components of objective tensors called respectively the GREEN-OLDROYD [29], the COTTER-RIVLIN [30] and the JAUMANN [31] stress rate tensors. There is also the TRUESDELL [32] stress rate tensor with components

$$\check{\sigma}_{ij}(t) = \frac{D\sigma_{ij}(t)}{Dt} + v_{m,m}(t)\,\sigma_{ij}(t) - v_{j,m}(t)\,\sigma_{im}(t) - v_{i,m}(t)\,\sigma_{mj}(t) \qquad (2.1.16)$$

formed from equation (2.1.15a) by addition of expression $v_{m,m}(t)\,\sigma_{ij}(t)$. In fact from any of the stress rates defined above others can be formed by addition of dimensionally correct terms which involve components of the strain rate tensor. If the stressed medium and applied forces undergo a rigid body motion ($d_{ij}(t) = 0$), all the stress rates tensors above reduce to the JAUMANN stress rate tensor. Moreover, if S is a co-rotational reference frame ($\omega_{ij}(t) = 0$, $v_i(t) = 0$) and the stressed medium and applied forces undergo a rigid body motion ($d_{ij}(t) = 0$) and if the stress field σ_{ij} is independent of time with respect to S, then the JAUMANN stress rate tensor vanishes. Being objective tensors, all the stress rate tensors mentioned vanish with respect to objectively equivalent reference frames for rigid body motions of the stressed material whose stress field is independent of time when referred to a co-rotational reference frame.

In view of the lack of a unique objective stress rate tensor,

PRAGER [33] considered the implication of using the various derivatives defined by equations (2.1.15a,b,c) and equation (2.1.16) in the constitutive equations of the theory of plasticity. PRAGER concluded that the JAUMANN derivative defined by equation (2.1.15c) is the most suitable for use in the constitutive equations of plasticity since for this derivative only does zero stress rate imply stationary stress invariants.

2.2 CONVECTED TIME DIFFERENTIATION

The stress rate tensors (2.1.15a,b,c) can be derived by a method attributed to CAUCHY [34] and generalized by OLDROYD [22]. Convected differentiation with respect to time t of a tensor intrinsically associated with the deforming material is introduced. This operation involves no dependence on a fixed frame of reference or on the motion of the material in space.

Restricting discussion to a tensor T with components T^{mn}_{ij} referred to a fixed curvilinear coordinate system x^i $(i = 1,2,3)$, the convected time derivative $\frac{\delta T}{\delta t}$ of T is defined as that tensor under all coordinate transformations which reduces to $\frac{D\overline{T}}{Dt}$ (or $\frac{\partial \overline{T}}{\partial t}$) in any convected coordinate system $\xi_i (i = 1,2,3)$ where $\overline{T}$ denotes the tensor T in the convected system and $\overline{T}^{\alpha\beta}_{\gamma\delta}$ denotes its components with respect to this system. T is assumed to be a tensor under all coordinate transformations, including transformations to convected coordinates. Thus

$$\overline{T}^{\alpha\beta}_{\gamma\delta} \frac{\partial x^m}{\partial \xi^\alpha} \frac{\partial x^n}{\partial \xi^\beta} = T^{mn}_{ij} \frac{\partial x^i}{\partial \xi^\gamma} \frac{\partial x^j}{\partial \xi^\delta} . \qquad (2.2.1)$$

Differentiating equation (2.2.1) with respect to time t holding convected coordinates constant, indicated by $\frac{D}{Dt}$, and using

$$\frac{D}{Dt}\left(\frac{\partial x^m}{\partial \xi^\alpha}\right) = \frac{\partial v^m}{\partial x^n}\frac{\partial x^n}{\partial \xi^\alpha} ,$$

one obtains

$$\frac{D\bar{T}^{uv}_{\gamma\delta}}{Dt} = \frac{\partial \xi^u}{\partial x^m}\frac{\partial \xi^v}{\partial x^n}\frac{\partial x^i}{\partial \xi^\gamma}\frac{\partial x^j}{\partial \xi^\delta}\left\{\frac{DT^{mn}_{ij}}{Dt} + T^{mn}_{pj}\frac{\partial v^p}{\partial x^i}\right.$$

$$\left. + T^{mn}_{ip}\frac{\partial v^p}{\partial x^j} - T^{pn}_{ij}\frac{\partial v^m}{\partial x^p} - T^{mp}_{ij}\frac{\partial v^n}{\partial x^p}\right\} . \qquad (2.2.2)$$

Since T^{mn}_{ij} are functions of x^i and t,

$$\frac{DT^{mn}_{ij}}{Dt} = \frac{\partial T^{mn}_{ij}}{\partial t} + \frac{\partial T^{mn}_{ij}}{\partial x^p} v^p .$$

Thus, from equation (2.2.2), it is seen that the components

$$\frac{\delta}{\delta t} T^{mn}_{ij} \equiv \frac{\partial T^{mn}_{ij}}{\partial t} + \frac{\partial T^{mn}_{ij}}{\partial x^p} v^p + T^{mn}_{pj}\frac{\partial v^p}{\partial x^i} + T^{mn}_{ip}\frac{\partial v^p}{\partial x^j}$$

$$- T^{pn}_{ij}\frac{\partial v^m}{\partial x^p} - T^{mp}_{ij}\frac{\partial v^n}{\partial x^p}$$

reduce to $\frac{D\bar{T}^{mn}_{ij}}{Dt}$ under transformations from the fixed spatial coordinate system to a convected coordinate system. The components $\frac{\delta}{\delta t} T^{mn}_{ij}$ are components of an absolute tensor since they are identically equal to the tensor components

$$\frac{\partial T^{mn}_{ij}}{\partial t} + T^{mn}_{ij,p}\, v^p + T^{mn}_{pj}\, v^p_{,i} + T^{mn}_{ip}\, v^p_{,j}$$

$$- T^{pn}_{ij}\, v^m_{,p} - T^{mp}_{ij}\, v^n_{,p}$$

where subscript commas denote covariant differentiation.

For the contravariant stress components σ^{mn},

$$\frac{\delta}{\delta t}\,\sigma^{mn} = \frac{\partial \sigma^{mn}}{\partial t} + \sigma^{mn}_{,p}\, v^p - \sigma^{pn}\, v^m_{,p} - \sigma^{mp}\, v^n_{,p} \qquad (2.2.4)$$

which with respect to a rectangular CARTESIAN coordinate system are

$$\frac{\delta}{\delta t}\,\sigma^{mn} = \frac{D\sigma_{mn}}{Dt} - \sigma_{pn}\, v_{m,p} - \sigma_{mp}\, v_{n,p},$$

the components of the GREEN-OLDROYD objective stress rate tensor. Similarly, for the covariant stress components σ_{mn} ,

$$\frac{\delta}{\delta t}\,\sigma_{mn} = \frac{\partial \sigma_{mn}}{\partial t} + \sigma_{mn,p}\, v^p + \sigma_{pn}\, v^p_{,m} + \sigma_{mp}\, v^p_{,n} \qquad (2.2.5)$$

which with respect to a rectangular CARTESIAN coordinate system are

$$\frac{\delta}{\delta t}\,\sigma_{mn} = \frac{D\sigma_{mn}}{Dt} + \sigma_{pn}\, v_{p,m} + \sigma_{mp}\, v_{p,n} ,$$

the components of the COTTER-RIVLIN objective stress rate tensor.

Let g_{mn} and g^{mn} be respectively the covariant and contravariant components of the metric tensor for a fixed spatial curvilinear coordinate system. Then since $Dg_{mn}/Dt = 0$ and $Dg^{mn}/Dt = 0$,

$$\frac{\delta}{\delta t} g^{mn} = - 2d^{mn} ,$$

$$\frac{\delta}{\delta t} g_{mn} = 2d_{mn} , \qquad (2.2.6a,b,c)$$

and

$$\frac{\delta}{\delta t} (g_{mn} \sigma^{np}) = \sigma^{np} \frac{\delta}{\delta t} g_{mn} + g_{mn} \frac{\delta}{\delta t} \sigma^{np} .$$

From equations (2.2.6b,c) it follows that

$$g_{mn} \frac{\delta}{\delta t} \sigma^{np} = \frac{\delta}{\delta t} (g_{mn} \sigma^{np}) - 2\sigma^{np} d_{mn} ,$$

showing that the raising and lowering of indices is not commutative with the convected time derivative operator $\frac{\delta}{\delta t}$ except when the stressed material executes only a rigid body motion ($d_{ij} = 0$) at P. On transforming equations (2.2.4) and (2.2.5) to a fixed spatial curvilinear coordinate system such that $d_{ij} = 0$ instantaneously with respect to this system, one gets

$$\frac{\delta}{\delta t} \sigma^{mn} = \frac{\partial}{\partial t} \sigma^{mn} + \sigma^{mn}_{,p} v^{p} - \sigma^{pn} \omega^{m}_{.p} - \sigma^{mp} \omega^{n}_{.p} ,$$

and (2.2.7a,b)

$$\frac{\delta}{\delta t} \sigma_{mn} = \frac{\partial}{\partial t} \sigma_{mn} + \sigma_{mn,p} v^{p} + \sigma_{pm} \omega^{p}_{.n} + \sigma_{mp} \omega^{p}_{.n}$$

where

$$\omega^{p}_{.m} = \frac{1}{2} g^{pn}(v_{n,m} - v_{m,n})$$

and the subscript commas denote covariant differentiation. Relationships (2.2.7a,b) define components of absolute tensors which reduce to the components of the JAUMANN stress rate tensor given by definition (2.1.15c) on choosing the spatial coordinate system as rectangular CARTESIAN. The JAUMANN derivative thus measures the time rate of change with respect to a set of coordinate axes instantaneously rotating about itself with the angular velocity of the material. For this reason, the JAUMANN derivative is referred to as the co-rotational time derivative. Historically, the co-rotational time derivative was first introduced by ZAREMBA [35]. In this thesis, however, it will be called the JAUMANN derivative as it is in current plasticity literature. Since the JAUMANN derivative does not depend upon the components of the strain rate tensor, it is the simplest of all the objective time derivatives. Moreover, it is the only objective time derivative which is commutative with the operation of raising and lowering indices.

For purposes of reference, relationship (2.2.7a) is rewritten as

$$\frac{\mathcal{D}\sigma^{i}_{j}}{\mathcal{D}t} = \frac{\partial\sigma^{i}_{j}}{\partial t} + \sigma^{i}_{j,k}\, v^{k} + \sigma^{i}_{k}\, \omega^{k}_{.j} - \sigma^{k}_{j}\, \omega^{i}_{.k} \; , \qquad (2.2.8)$$

which upon expanding the covariant derivatives becomes

$$\frac{\mathcal{D}\sigma^{i}_{j}}{\mathcal{D}t} = \frac{\partial\sigma^{i}_{j}}{\partial t} + \left(\frac{\partial\sigma^{i}_{j}}{\partial x^{k}} + \Gamma^{i}_{mk}\sigma^{m}_{j} - \Gamma^{m}_{jk}\sigma^{i}_{m}\right) v^{k} + \sigma^{i}_{k}\, \omega^{k}_{.j} - \sigma^{k}_{j}\, \omega^{i}_{.k} \qquad (2.2.9)$$

with

$$\Gamma^{i}_{jk} \equiv \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial x^{k}} + \frac{\partial g_{km}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{m}}\right) ,$$

the CHRISTOFFEL symbols of the second kind.

2.3 PRANDTL-REUSS EQUATIONS AT FINITE STRAIN

Following THOMAS [36], the PRANDTL-REUSS equations for plastic regions valid for finite strain are obtained by insuring that the classical PRANDTL-REUSS equations do not violate the 'principle of material objectivity'. This is accomplished simply by replacing the non-objective material derivative of stress deviation with the JAUMANN derivative of stress deviation. The strain rate tensor is not replaced since it is a measure of the instantaneous rate of change of lengths and angles of material elements in the deforming material [37] and is an objective tensor. Thus from equations (1.5.4a,b), together with

$$\frac{\mathcal{D}p}{\mathcal{D}t} = \frac{Dp}{Dt} ,$$

one obtains

$$d_{ij} = \frac{1}{2\mu} \frac{\mathcal{D}s_{ij}}{\mathcal{D}t} - \frac{1}{3K} \frac{Dp}{Dt} \delta_{ij} + \lambda s_{ij} , \quad \lambda = \frac{f^{m}_{n} s^{n}_{m}}{2k^{2}} ,$$

and (2.3.1a,b)

$$d^{k}_{k} = - \frac{1}{K} \frac{Dp}{Dt} ,$$

the constitutive equations for a compressible elastic-perfectly plastic MISES solid in finite strain. These reduce to

$$d_{ij} = \frac{1}{2\mu}\frac{\mathcal{D}s_{ij}}{\mathcal{D}t} + \lambda s_{ij} \ , \ \lambda = \frac{d^m_n s^n_m}{2k^2} \ ,$$

and (2.3.2a,b)

$$d^k_k = 0$$

for an incompressible elastic-perfectly plastic MISES solid. Similarly from equations (1.5.8a,b), one obtains

$$d_{ij} = \frac{1}{2\mu}\frac{\mathcal{D}s_{ij}}{\mathcal{D}t} - \frac{1}{3K}\frac{Dp}{Dt}\delta_{ij} + \sum_r \lambda_r \frac{\partial f_r}{\partial s_{ij}} \ ,$$

and (2.3.3a,b)

$$d^k_k = -\frac{1}{K}\frac{Dp}{Dt} \ ,$$

the constitutive equations for a compressible elastic-perfectly plastic TRESCA solid in the plastic range. Equations (2.3.3a,b) are also valid for an elastic-perfectly plastic solid when the yield surface has any finite number of surfaces with piecewise continuously turning normals.

For elastic regions, the constitutive equations valid for finite strain are obtained by employing proper objective tensors in equations (1.2.3a,b). This gives

$$\frac{\mathcal{D}s_{ij}}{\mathcal{D}t} = 2\mu\ f_{ij}\ ,$$

and (2.3.4a,b)

$$\frac{Dp}{Dt} = -\ K\ d^k_k\ ,$$

the PRANDTL-REUSS equations for an elastic solid.

By adopting the constitutive equations for a hypo-elastic solid, A.E. GREEN [38] also established the PRANDTL-REUSS equations at finite strain for a compressible elastic-perfectly plastic MISES solid. GREEN employed the GREEN-OLDROYD stress rate tensor and derived the constitutive equations

$$\frac{\delta}{\delta t}\ s^{ij} + s^{mj}\ f^i_m - s^{im}\ f^j_m = 2\mu\ f^{ij}\ ,$$

and (2.3.5a,b)

$$\frac{\delta}{\delta t}\ p = -\ K\ d^k_k\ ,$$

for the non-plastic region where $s^i_j\ s^j_i < 2k^2$, and

$$\frac{\delta}{\delta t}\ s^{ij} + s^{mj}\ f^i_m - s^{im}\ f^j_m = f^{ij} - \frac{f^m_n s^n_m}{2k^2}\ s^{ij}\ , \qquad (2.3.6a)$$

and

$$\frac{\delta}{\delta t} p = - K d_k^k , \qquad (2.3.6b)$$

for the plastic region where $s_j^i s_i^j = 2k^2$ and $d_j^i s_i^j > 0$. If the MISES solid is incompressible, equations (2.3.5a,b) and (2.3.6a,b) reduce to equations (2.3.4a,b) and (2.3.2a,b) respectively. A discussion of the relationship between a hypo-elastic material of grade two and an elastic-perfectly plastic MISES solid is found in CHAPTER V.

To eliminate the use of separate sets of constitutive equations in both the elastic and plastic regions, THOMAS [18] proposed a single set of constitutive equations valid for both regions. With the assumption that the constitutive equations have the form

$$\frac{\mathcal{D}\sigma_{ij}}{\mathcal{D}t} = A_{ij}^{mn} d_{mn} + B_{ij}^{mn} \sigma_{mn} ,$$

with A_{ij}^{mn} and B_{ij}^{mn} isotropic tensor functions of the stress deviation invariants and strain rate deviation invariants, together with plastic incompressibility and the VON MISES yield criterion, THOMAS established the constitutive equations

$$\frac{\mathcal{D}s_{ij}}{\mathcal{D}t} = 2\mu \{f_{ij} - [\frac{f_n^m s_m^n}{2k^2} + \gamma(1 - \frac{s_n^m s_m^n}{2k^2})] s_{ij}\}$$

and (2.3.7a,b)

$$\frac{\mathcal{D}\sigma_k^k}{\mathcal{D}t} = \frac{h\, d_k^k}{1 - (\frac{s_n^m s_m^n}{2k^2})} + b\, \sigma_k^k .$$

In general γ, h, b are invariants dependent on position and time but for homogeneous material, they must be material constants. For plastic flow, equations (2.3.7a,b) are such as to reduce to equations (2.3.2a,b).

2.4 PRANDTL-REUSS CONSTITUTIVE EQUATIONS AND FIELD EQUATIONS IN CURVILINEAR COORDINATES FOR AN INCOMPRESSIBLE ELASTIC-PERFECTLY PLASTIC MISES SOLID IN FINITE STRAIN

Let $x^i (i = 1,2,3)$ be a fixed curvilinear coordinate system with g^{ij} and g_{ij} the contravariant and covariant components respectively of the metric tensor. Then the governing equations for the plastic flow of an incompressible elastic-perfectly plastic MISES solid with respect to this coordinate system with commas denoting covariant differentiation with respect to the metric are:

(i) the PRANDTL-REUSS equations (2.3.2a), namely,

$$\frac{\mathcal{D}s^i_j}{\mathcal{D}t} = 2\mu \, (d^i_j - \lambda s^i_j) \tag{2.4.1a}$$

with

$$\lambda = \frac{1}{2k^2} d^m_n s^n_m$$

and

$$\frac{\mathcal{D}s^i_j}{\mathcal{D}t} \equiv \frac{\partial s^i_j}{\partial t} + (\frac{\partial s^i_j}{\partial x^k} + \Gamma^i_{mk} \, s^m_j - \Gamma^m_{jk} \, s^i_m) \, v^k + s^i_k \, \omega^k_{\cdot j} - s^k_j \, \omega^i_{.k}$$

as obtained from equation (2.2.10) on replacing σ^i_j by s^i_j, and the

equation of incompressibility

$$d^i_i = 0\ , \tag{2.4.1b}$$

where

$$d^i_j = \frac{1}{2}\, g^{ik}(v_{k,j} + v_{j,k})$$

and v^i are the contravariant components of the velocity vector;

(ii) the equations of motion

$$\sigma^{ij}_{\ ,j} + \rho g^{ij}\, F_j = \rho\left(\frac{\partial v^i}{\partial t} + v^i_{\ ,j}\, v^j\right) \tag{2.4.2}$$

where

$$\sigma^{ij} = s^{ij} - pg^{ij},\ p = -\frac{1}{3}(g_{mn}\,\sigma^{mn})\ ,$$

ρ is the density, and F_i the covariant components of the body force per unit mass;

(iii) the equation of continuity

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0\ , \tag{2.4.3}$$

(iv) the VON MISES yield criterion

$$s^i_j s^j_i = 2k^2\ , \tag{2.4.4}$$

and

(v) the identity

$$s^i_i \equiv 0 \ . \qquad (2.4.5)$$

In general, the governing equations must be used to determine eleven unknowns; namely, the six components s^{ij} of the symmetric stress deviation tensor, the three velocity components v^i, the density ρ, and the hydrostatic pressure $-p$. As listed, the governing equations consist of eleven quasi-linear partial differential equations (2.4.1a), (2.4.1b), (2.4.2) and (2.4.3) and two finite equations (2.4.4) and (2.4.5). However, equations (2.4.1a) are not independent of the two finite equations since on taking the JAUMANN derivative of both sides of equations (2.4.4) and (2.4.5), one obtains

$$s^i_j \frac{\mathcal{D}s^j_i}{\mathcal{D}t} = 0 \text{ and } \frac{\mathcal{D}s^i_i}{\mathcal{D}t} = 0,$$

showing that there are only four independent PRANDTL-REUSS equations. Thus there are available nine partial differential equations and two finite equations for the determination of the eleven unknowns.

For quasi-static, incompressible elastic-perfectly plastic problems in the absence of body forces, the stress and velocity field can be determined with ten of the governing equations; namely, equations (2.4.1a,b), (2.4.4), (2.4.5) and the equations of equilibrium

$$\sigma^{ij}_{,j} = 0.$$

The density ρ, if required, is determined by equation (2.4.3). The discussion also applies if the flow should be steady.

If in the general case the unknowns are taken as the six independent components σ^{ij} of the symmetric stress tensor, the three velocity components v^i and the density ρ, then there are available four independent PRANDTL-REUSS equations, three equilibrium equations, one incompressibility equation, the continuity equation and the yield condition for the determination of these ten unknowns.

For the general cases considered above, THOMAS [39] remarks that, as an alternative, five PRANDTL-REUSS equations may be used provided the VON MISES yield criterion is not used directly but only as a boundary condition on the surface S enveloping the plastic deforming region.

In APPENDIX A of this thesis, the PRANDTL-REUSS equations (2.4.1a) are expressed in terms of the physical components of the stress deviation tensor and the velocity vector with respect to a spherical polar coordinate system (r,θ,ϕ) and a cylindrical polar coordinate system (r,θ, z).

CHAPTER III

THE PLANE RADIAL FLOW OF AN INCOMPRESSIBLE ELASTIC-PERFECTLY PLASTIC MISES SOLID

A stress field for plane flow of an isotropic rigid-perfectly plastic solid through a converging channel with rough sides was obtained by NADAI [40]. Later HILL [41] obtained a kinematically admissible velocity field associated with this stress field. The streamlines of this velocity field are radii directed through the virtual apex of the channel and the components of the stress deviation tensor of the stress field are independent of the radial distance from the virtual apex. In this chapter an investigation is undertaken of the plane radial flow of an incompressible elastic-perfectly plastic MISES solid through a converging channel with perfectly rough sides, that is, sides on which the frictional stress is equal to the shear yield stress. The case when the assumed constant frictional stress developed on the channel walls is less than the shear yield stress is also considered.

The solution of NADAI and HILL is valid for a rigid-perfectly plastic solid with either the TRESCA or the VON MISES yield criterion provided the appropriate value of the shear yield stress is used. However, this investigation is restricted to a MISES solid, that is, one whose yield is governed by the VON MISES yield criterion.

3.1 PLANE RADIAL FLOW THROUGH CONVERGING CHANNEL

Referring to Figure 2, let (r, θ, z) be cylindrical polar coordinates and let the channel occupy the region $-\alpha \leq \theta \leq \alpha$ with the flow parallel to the plane $z = 0$. The flow is steady and quasi-

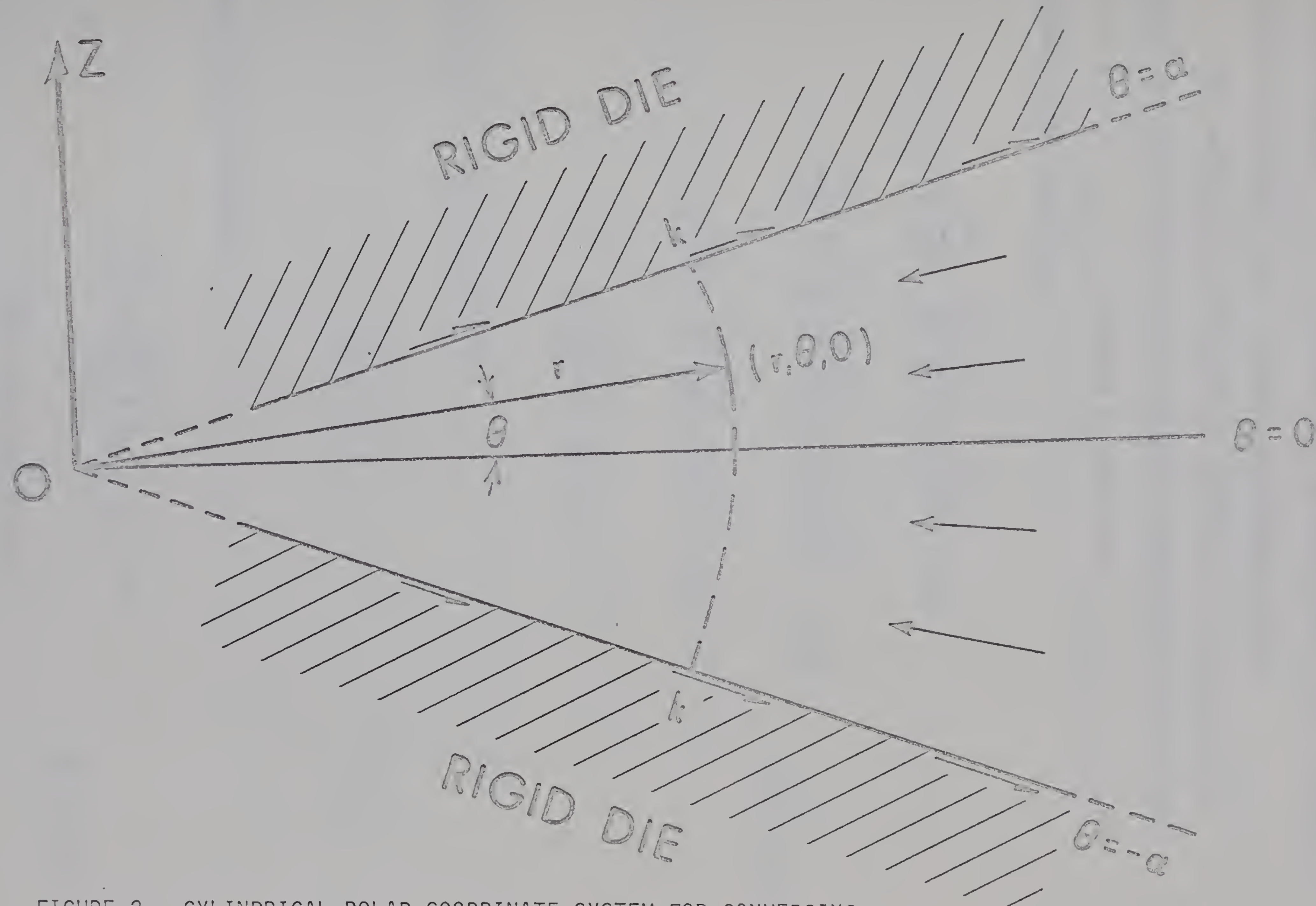


FIGURE 2 - CYLINDRICAL POLAR COORDINATE SYSTEM FOR CONVERGING FLOW THROUGH A CONVERGING CHANNEL

static with stream-lines that are radii passing through the virtual apex of the channel. It is assumed that the VON MISES yield criterion is satisfied throughout the field and no unloading occurs.

With reference to APPENDIX A, SECTION A.4, the governing equations in terms of the non-zero physical components of the stress deviation s_r, s_θ, $s_{r\theta}$, s_z, velocity v_r and stress σ_r, σ_θ, $\tau_{r\theta}$, σ_z are the PRANDTL-REUSS equations

$$
\begin{aligned}
&v_r \frac{\partial s_r}{\partial r} - \frac{s_{r\theta}}{r} \frac{\partial v_r}{\partial \theta} = 2\mu\left(\frac{\partial v_r}{\partial r} - \lambda s_r\right), \\
&v_r \frac{\partial s_{r\theta}}{\partial r} + \frac{s_r - s_\theta}{2r} \frac{\partial v_r}{\partial \theta} = 2\mu\left(\frac{1}{2r} \frac{\partial v_r}{\partial \theta} - \lambda s_{r\theta}\right), \\
&v_r \frac{\partial s_\theta}{\partial r} + \frac{s_{r\theta}}{r} \frac{\partial v_r}{\partial \theta} = 2\mu\left(\frac{v_r}{r} - \lambda s_\theta\right), \\
&v_r \frac{\partial s_z}{\partial r} = -2\mu\, \lambda s_z ;
\end{aligned}
\tag{3.1.1a,b,c,d}
$$

the equation of incompressibility

$$
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 ; \tag{3.1.2}
$$

the equilibrium equations

$$
\begin{aligned}
&r \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + (\sigma_r - \sigma_\theta) = 0 , \\
&r \frac{\partial \tau_{r\theta}}{\partial r} + 2\tau_{r\theta} + \frac{\partial \sigma_\theta}{\partial \theta} = 0 ;
\end{aligned}
\tag{3.1.3a,b}
$$

the yield criterion

$$s_r{}^2 + s_\theta{}^2 + s_z{}^2 + 2s_{r\theta}{}^2 = 2k^2 \; ; \tag{3.1.4}$$

and the identity

$$s_r + s_\theta + s_z \equiv 0 \; .$$

Equation (3.1.2) requires that

$$v_r = \frac{g(\theta)}{r} \tag{3.1.5}$$

and this gives the following non-zero components of strain rate:

$$d_r = -\frac{g(\theta)}{r^2} \; , \; d_\theta = \frac{g(\theta)}{r^2} \; , \; d_{r\theta} = \frac{1}{2}\,\frac{g'(\theta)}{r^2} \; . \tag{3.1.6}$$

Equations (3.1.6) show that ratios of the strain rate components do not vary along a radius. The considered solid being elastically and plastically incompressible, both the elastic and plastic components of velocity satisfy the incompressibility equation (3.1.2) and the uniquely determined physical components of the elastic and plastic strain rate tensor are of the form given by equations (3.1.6). Since

$$s^i_j = \frac{\sqrt{2}k d^{i(P)}_j}{\sqrt{d^{m(P)}_n \; d^{n(P)}_m}}$$

by equation (1.3.9b), s^i_j must be functions of θ alone. It follows then, from equation (3.1.1d), that $\lambda s_z = 0$ and since λ is non-zero

where flow occurs

$$s_z = -(s_r + s_\theta) = 0.$$

Substitution of $s_\theta = - s_r$ in the yield criterion (3.1.4) gives

$$s_r^{\ 2} + s_{r\theta}^{\ \ 2} = k^2 \ . \qquad (3.1.7)$$

Equations (3.1.3a,b) and (3.1.7) are also valid for the corresponding flow of a rigid-perfectly plastic solid and the stress boundary conditions $\tau_{r\theta} = k$ at $\theta = \alpha$ and $\tau_{r\theta} = k$ at $\theta = -\alpha$ are the same. Consequently, the stress field is identical to that obtained by NADAI.

3.2 DETERMINATION OF STRESS FIELD

NADAI'S approach [40] for determining the stress field is as follows. The stress field is expressed in terms of the angle ψ between the direction of the algebraically greatest principal stress and a radius. The angle ψ is a function of θ only and it has the same sign as θ since the friction acts so as to oppose the relative motion or tendency for relative motion between the solid and the channel walls; ψ is zero on the axis $\theta = 0$ and ranges between $-\pi/4$ and $\pi/4$. The yield criterion (3.1.7) is parametrized in terms of ψ by letting

$$s_{r\theta} = k \sin 2\psi$$

and (3.2.1a,b)

$$s_r = - s_\theta = k \cos 2\psi \ .$$

From the equation of equilibrium (3.1.3b),

$$\frac{\partial \sigma_\theta}{\partial \theta} = - 2k \sin 2\psi$$

from which

$$\sigma_\theta = - 2k \int_0^\theta \sin 2\psi \, d\theta + h(r)$$

and, hence from equation (3.2.1b),

$$\sigma_r = 2k \cos 2\psi - 2k \int_0^\theta \sin 2\psi \, d\theta + h(r). \qquad (3.2.2)$$

Substitution of equations (3.2.1a,b) and (3.2.2) into the equation of equilibrium (3.1.3a) gives

$$2k \cos 2\psi \frac{d\psi}{d\theta} + 2k \cos 2\psi = - r \frac{dh}{dr} .$$

Setting each side of this equation equal to a constant 2ck yields

$$\frac{dh}{dr} = - \frac{2ck}{r} ,$$

and (3.2.3a,b)

$$\frac{d\theta}{d\psi} = \frac{\cos 2\psi}{c - \cos 2\psi} .$$

Integration of these differential equations yields

$$h(r) = -2ck\,\ell n r + A_o ,$$

and

$$\theta = -\psi + \frac{c}{\sqrt{c^2 - 1}} \tan^{-1} \left(\sqrt{\frac{c + 1}{c - 1}} \tan \psi\right) , \qquad (3.2.4)$$

where A_o is an arbitrary constant and, since $\psi = \pi/4$ on $\theta = \alpha$, c is given by

$$\frac{c}{\sqrt{c^2 - 1}} \tan^{-1} \left(\sqrt{\frac{c + 1}{c - 1}}\right) = \frac{\pi}{4} + \alpha. \qquad (3.2.5)$$

Also, c varies from ∞ to 1.1922 as α varies from 0 to $\pi/2$. Now

$$\int_0^{\theta} \sin 2\psi \, d\theta = \int_0^{\psi} \sin 2\psi \frac{\cos 2}{c - \cos 2\psi} \, d\psi$$

$$= \frac{1}{2} \cos 2\psi - \frac{1}{2} + \frac{c}{2} \ell n \frac{c - \cos 2\psi}{c - 1}$$

and, hence

$$p = -\sigma_z = -\frac{1}{2} (\sigma_r + \sigma_\theta)$$

$$= 2k \left[c\ell n\, r + \frac{c}{2} \ell n\, (c - \cos 2\psi) + A\right] \qquad (3.2.6)$$

where A is an arbitrary constant. Equations (3.2.1a,b) and (3.2.6) completely determine the stress field to within an arbitrary hydrostatic pressure.

3.3 DETERMINATION OF VELOCITY FIELD

Elimination of λ between equations (3.1.1b) and (3.1.1c) and substitution of equation (3.1.5) into the result gives

$$\frac{g'(\theta)}{g(\theta)} = \frac{2\mu s_{r\theta}}{s_{r\theta}^{2} - \mu s_{r} + s_{r}^{2}} \, . \tag{3.3.1}$$

Substitution of the components of the stress deviation given by equations (3.2.1a,b) into equation (3.3.1) gives

$$\frac{g'(\theta)}{g(\theta)} = \frac{2\mu \sin 2\psi}{k - \mu \cos 2\psi} \, . \tag{3.3.2}$$

With the use of equation (3.2.4b), this becomes

$$\frac{dg(\theta)}{g(\theta)} = \frac{2 \sin 2\psi \cos 2\psi \, d\psi}{(n - \cos 2\psi)(c - \cos 2\psi)}$$

where $n \equiv \frac{k}{\mu} \geq 0$ and hence, provided $\cos 2\psi \neq n$, integration gives

$$g(\theta) = D \left[\frac{|\cos 2\psi - n|^{n}}{(c - \cos 2\psi)^{c}}\right]^{1/c-n} \tag{3.3.3}$$

where D is the constant of integration. For $n = 0$,

$$g(\theta) = \frac{D}{c - \cos 2\psi}$$

which is the expression obtained by HILL [41] for the rigid-perfectly plastic solid. From equations (3.1.5) and (3.3.3),

$$v_r = \frac{D}{r}\left[\frac{|\cos 2\psi - n|^n}{(c - \cos 2\psi)^c}\right]^{\frac{1}{c-n}} , \quad n \neq \cos 2\psi , \qquad (3.3.4)$$

where c is given by equation (3.2.6) and D is obtained from the volume flow per unit thickness and is negative since v_r is negative.

The scalar invariant λ, expressed in terms of the physical components of the strain rate tensor and stress deviation tensor, is

$$\lambda = \frac{1}{2k^2} (d_r s_r + 2\, d_{r\theta} s_{r\theta} + d_\theta s_\theta) .$$

Substituting equations (3.1.6) and (3.1.8a) gives

$$\lambda = \frac{1}{2k^2} \frac{g(\theta)}{r^2} [s_\theta - s_r + \frac{g'(\theta)}{g(\theta)} s_{r\theta}] ,$$

and substituting equations (3.2.1a,b) and (3.3.2) gives

$$\lambda = \frac{v_r}{kr} \left(\frac{1 - n \cos 2\psi}{n - \cos 2\psi}\right) . \qquad (3.3.5)$$

For plastic flow, λ is positive.

If $n \equiv k/\mu < 1$, then according to equation (3.3.4), $\lim_{\theta \to \pm\theta_o} v_r = 0$ where θ_o is the value of θ corresponding to $2\psi = \cos^{-1} n$. In the region $-\theta_o < \theta < \theta_o$, the determined velocity field and the stress field satisfy the governing equations and $\lambda > 0$. However, in the regions $-\alpha \leq \theta < -\theta_o$ and $\theta_o < \theta \leq \alpha$ the velocity field and the stress field result in $\lambda < 0$. To obtain a positive λ it is necessary to take $D > 0$, but then the friction shearing stresses $\tau_{r\theta} = k$ at $\theta = \alpha$ and $\tau_{r\theta} = -k$ at $\theta = -\alpha$ do not oppose

the flow. Consequently, the velocity field given by equation (3.3.4) is not admissible in the regions $-\alpha \leq \theta < -\theta_o$ and $\theta_o < \theta \leq \alpha$ unless D is zero. From equations (3.3.2) and (3.3.4), one obtains that

$$g'(\theta) = \frac{-D \sin 2\psi}{(c - \cos 2\psi)^{\frac{c}{c-n}}} (\cos 2\psi - n)^{\frac{2n-c}{c-n}} , \quad D < 0 ,$$

in the region $-\theta_o < \theta < \theta_o$ where $n < \cos 2\psi \leq 1$. Thus for $c < 2n$, $\lim_{\theta \to \pm\theta_o} g'(\theta) = 0$ and the velocity profile is normal to the radii $\theta = \pm\theta_o$. For $c = 2n$, $\lim_{\theta \to \pm\theta_o} g'(\theta) = -2D/n^2 (1 - n^2)^{1/2} > 0$. For $c > 2n$, $\lim_{\theta \to \pm\theta_o} g'(\theta) = +\infty$ and the velocity profile is tangential to the radii $\theta = \pm\theta_o$. Also, infinite values of $d_{r\theta}$ result as the radii $\theta = \pm\theta_o$ are approached from the region $-\theta_o < \theta < \theta_o$; this is permissible since the material considered is not viscous. Moreover, the parameter λ undergoes an infinite discontinuity at $\theta = \pm\theta_o$ but this is not inconsistent with the constitutive equations (3.1.1).

If $n = 1$, equation (3.3.5) reduces to

$$\lambda = \frac{v_r}{kr}$$

from which it follows that $v_r = 0$ for $-\alpha \leq \theta \leq \alpha$ is the only admissible velocity field.

If $n > 1$, then for $\lambda > 0$ the velocity field is such that the radial velocity decreases as θ varies laterally along a θ - line from $\theta = \pm\theta_o$ to $\theta = 0$; θ_o is the value of θ corresponding to $2\psi = \cos^{-1}\frac{1}{n}$. This is physically unlikely; moreover, since all common elastic-plastic materials have small values of $n < 1$, the phenomenon is of no great practical importance.

3.4 CONCLUSION

For the region $-\theta_o < \theta < \theta_o$, the solution to the converging radial flow problem involving a channel with perfectly rough sides is provided by the velocity equation (3.3.4) and by equations (3.2.1a,b) and equation (3.2.7) which determine the stress field to within an arbitrary hydrostatic pressure. For the regions $-\alpha \leq \theta \leq -\theta_o$ and $\theta_o \leq \theta \leq \alpha$, the solution is provided by $v_r = 0$ and the same stress equations. These solutions indicate that the velocity v_r is zero at the channel walls whereas the velocity field obtained by HILL for the corresponding problem involving a rigid-perfectly plastic solid indicates that there is slip at the channel walls. If $n \ll 1$, the angle $\alpha - \theta_o$ is small compared with α as shown by the values in the following table. These values are for the particular case of $\alpha = 24°17'$ $(c = 2)$.

TABLE 1

VARIATION OF $\alpha - \theta_o$ WITH $n \equiv \frac{k}{\mu}$

$[\alpha = 24°17'$ $(c = 2)]$

n	$\alpha - \theta_o$
0.001	0°0'4.5"
0.01	0°0'12.6"
0.05	0°1'10"
0.1	0°4'40"

Figure 3 shows $g(\theta)$ for an elastic-perfectly plastic solid with $n = 0.1$ and $D = -1$ and also for an rigid plastic solid for $\alpha = 24°17'$ $(c = 2)$ and $D = -1$. For smaller values of n, of the order of 10^{-3}, which are realistic for metals, the difference between the

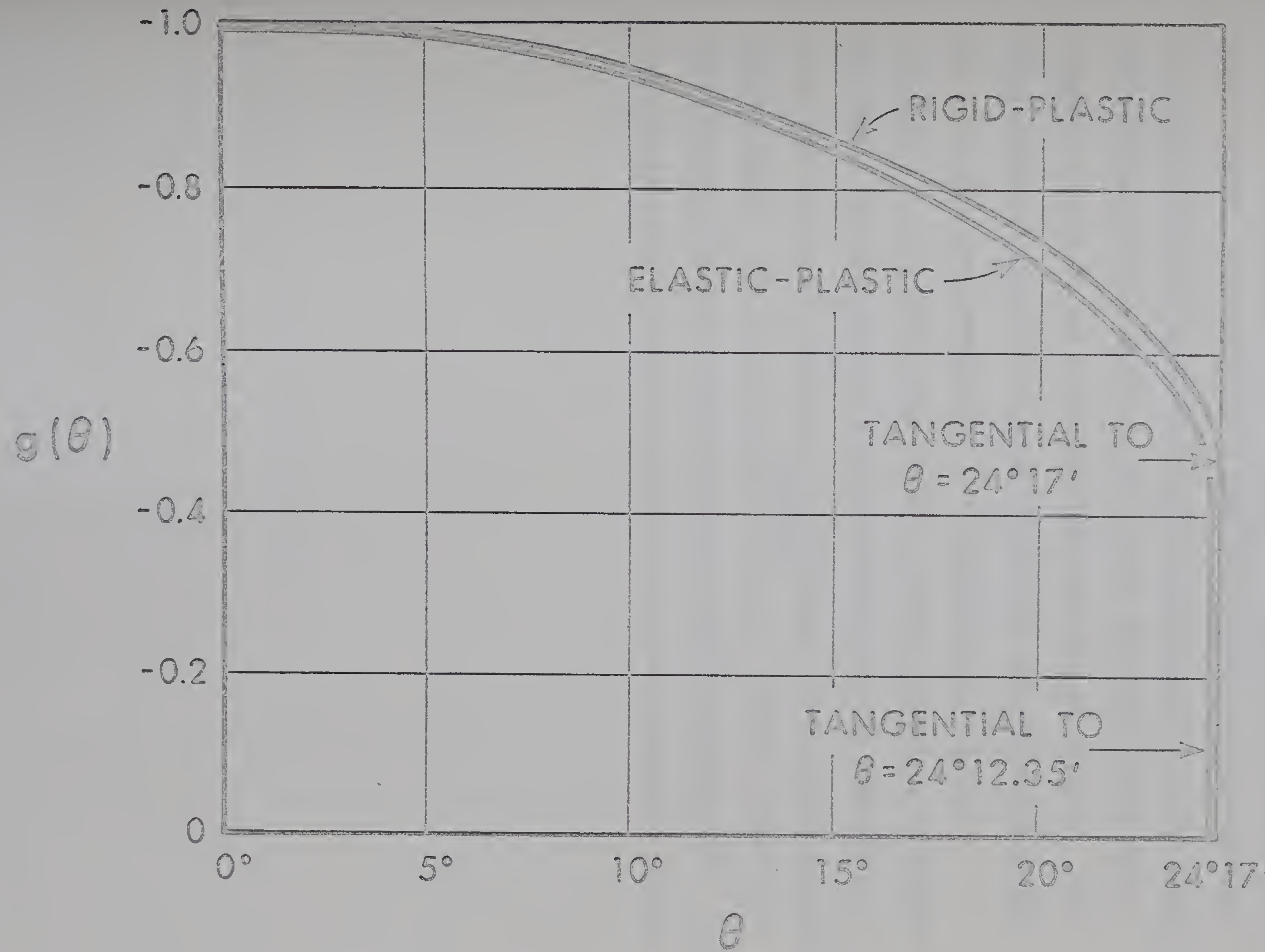


FIGURE 3 - GRAPH OF $g(\theta)$ FOR $\alpha = 24°17'$ ($c = 2$) AND $D = -1$

velocity profiles for the elastic-perfectly plastic and rigid-perfectly plastic solid is negligible except near the sides.

From the present solution involving the elastic-perfectly plastic solid, it is possible to find the solution when the constant frictional stress on the channel walls is mk ($0 \leq m \leq 1$). The stress distribution within a sector $\pm \theta (\theta \leq \alpha)$ is the stress field for a channel of angle 2θ and a constant frictional stress $k \sin 2\psi$. Thus to find the solution for a channel of angle 2θ and frictional stress mk, it is required to determine c from equation (3.2.4) with $\psi = \frac{1}{2} \sin^{-1} m$. Then the stress field is given by equations (3.2.1a,b) and (3.2.6) and the velocity field by equation (3.3.4) provided m is such that $0 \leq m < \sqrt{1 - n^2}$, $n < 1$, in which case there is slip along the channel walls. For m such that $\sqrt{1 - n^2} \leq m \leq 1$, there is developed a non-deforming region and there is no slip along the channel walls. In the deforming region, the velocity field is given by equation (3.3.4).

CHAPTER IV

ELASTIC-PERFECTLY PLASTIC FLOW THROUGH A CONVERGING CONICAL CHANNEL

SHIELD [42] considered the axially-symmetric flow of a rigid-perfectly plastic material forced through a rigid conical channel, outlined the method of solution for a general yield criterion and gave solutions using the VON MISES and TRESCA yield criteria. These solutions involve streamlines that are radii passing through the virtual apex of the cone and a constant friction stress between the channel wall and the material is assumed. The corresponding problem with an incompressible elastic-perfectly plastic MISES solid is considered in this chapter.

Axially symmetric flow through the converging channel is considered and the channel is assumed sufficiently long that the inlet and exit effects can be neglected. It is further assumed that the flow is steady and quasi-static and that all points in the stress field satisfy the VON MISES yield criterion and that no unloading is taking place. A solution with streamlines that are radii directed through the virtual apex of the channel is sought.

4.1 GOVERNING EQUATIONS FOR RADIAL FLOW

Let (r,θ,ϕ) be spherical polar coordinates with the axis of the channel given by $\theta = 0$ and the conical surface by $\theta = \alpha$ as illustrated in Figure 4. It is assumed that the friction stress acting on the material at the surface of the channel $\theta = \alpha$ is $\tau_{r\theta} = mk$ where m is a constant. Since k is the shear yield stress and $\tau_{r\theta}$ is non-negative

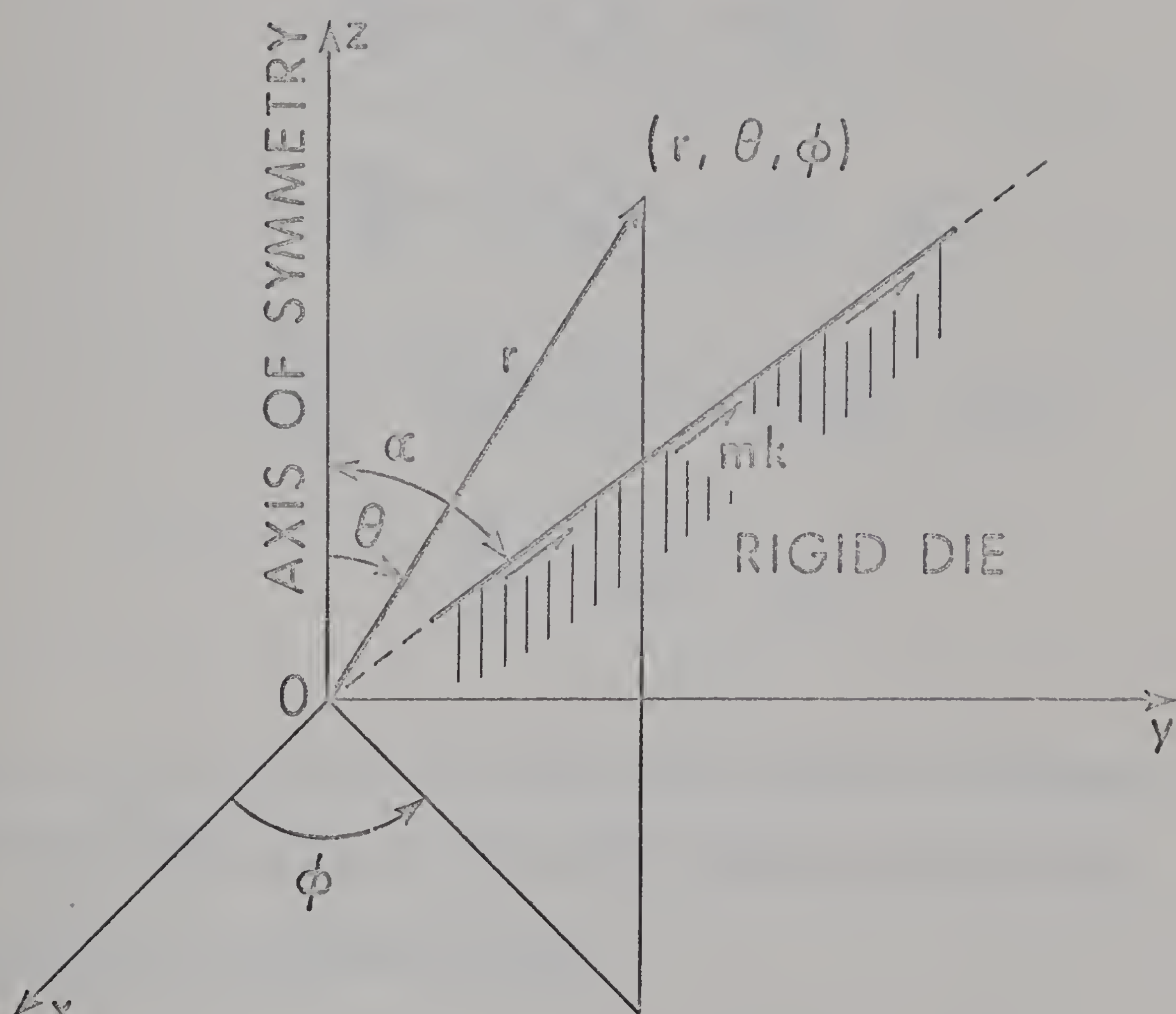


FIGURE 4 - SPHERICAL POLAR COORDINATE SYSTEM FOR CONVERGING FLOW THROUGH A CONICAL CHANNEL

for converging flow, m must satisfy $0 \leq m \leq 1$.

With reference to APPENDIX A, SECTION A.3, the governing equations for axially symmetric flow, with the assumption that the velocity component v_θ is zero, are

the PRANDTL-REUSS equations

$$v_r \frac{\partial s_r}{\partial r} - \frac{s_{r\theta}}{r} \frac{\partial v_r}{\partial \theta} = 2\mu \left[\frac{\partial v_r}{\partial r} - \lambda s_r\right] ,$$

$$v_r \frac{\partial s_\theta}{\partial r} + \frac{s_{r\theta}}{r} \frac{\partial v_r}{\partial \theta} = 2\mu \left[\frac{v_r}{r} - \lambda s_\theta\right],$$

$$v_r \frac{\partial s_{r\theta}}{\partial r} + \frac{s_r - s_\theta}{2r} \frac{\partial v_r}{\partial \theta} = 2\mu \left[\frac{1}{2r} \frac{\partial v_r}{\partial \theta} - \lambda s_{r\theta}\right] , \qquad (4.1.1a,b,c,d)$$

$$v_r \frac{\partial s_\phi}{\partial r} = 2\mu\left[\frac{v_r}{r} - \lambda s_\phi\right] ,$$

where s_r, s_θ, s_ϕ and $s_{r\theta}$ are the non-zero physical components of the stress deviation and v_r is the radial component of velocity;

the equation of incompressibility

$$\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} = 0 \; ; \qquad (4.1.2)$$

the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r} (2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{r\theta} \cot \theta) = 0,$$

$$(4.1.3a,b)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r} [(\sigma_\theta - \sigma_\phi) \cot \theta + 3\tau_{r\theta}] = 0;$$

and the VON MISES condition

$$(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_\phi)^2 + (\sigma_\phi - \sigma_r)^2 + 6\tau_{r\theta}{}^2 = 6k^2, \quad (4.1.4)$$

where σ_r, σ_θ, σ_ϕ and $\tau_{r\theta}$ are the non-zero physical components of the stress tensor. Only three of the PRANDTL-REUSS equations are independent since the sum of equations (4.1.1a), (4.1.1b) and (4.1.1d) is identically zero. Consequently, there are seven independent equations for the six unknowns s_r, s_θ, s_ϕ, $s_{r\theta}$, p and v_r. The prior assumption that $v_\theta = 0$ is the reason for the additional equation. If a solution with $v_\theta = 0$ can be found which satisfies the governing equations (4.1.1) to (4.1.4), the prior assumption is justified.

Equation (4.1.2) requires that

$$v_r = \frac{Bg(\theta)}{r^2} \qquad (4.1.5)$$

where $g(\theta)$ is a function of θ only and B is a constant. The components of the strain rate tensor are then

$$d_r = -2B\,\frac{g(\theta)}{r^3}, \quad d_\phi = B\,\frac{g(\theta)}{r^3}, \quad d_\theta = B\,\frac{g(\theta)}{r^3}, \quad d_{r\theta} = \frac{B}{2}\,\frac{g'(\theta)}{r^3}.$$

The ratios of the strain rate components obtained from equation (4.1.5) do not vary along a radius. Therefore, following the same argument given in the two dimensional plane problem, the components of the stress deviation tensor can be shown to be functions of θ only.

Consequently, eliminating λ from equations (4.1.1a) and (4.1.1c) gives

$$\frac{g'(\theta)}{g(\theta)} = \frac{4\mu\ s_{r\theta}}{[s_{r\theta}^{2} - \frac{s_r}{2}\ (2\mu - s_r + s_\theta)]}\ , \tag{4.1.6}$$

and from equations (4.1.1a) and (4.1.1b), together with use of equation (4.1.2),

$$\frac{g'(\theta)}{g(\theta)} = \frac{2\mu}{s_{r\theta}}\ [\frac{s_r + 2s_\theta}{s_r + s_\theta}]. \tag{4.1.7}$$

Equating the right hand sides of equations (4.1.6) and (4.1.7) and simplifying gives

$$s_{r\theta}^{2} = \frac{1}{2}\ s_r s_\theta + \frac{1}{2}\ s_r^{2} - s_\theta^{2} - \mu s_r - 2\mu s_\theta. \tag{4.1.8}$$

Equation (4.1.8) may be rewritten as

$$6s_{r\theta}^{2} = 3(s_r + s_\theta)^2 - 3(s_r + 3s_\theta)\ s_\theta - 6\mu\ (s_r + 2s_\theta). \tag{4.1.9}$$

Combining equation (4.1.9) with the yield criterion (4.1.4) gives

$$\begin{aligned}(s_r - s_\theta)^2 + (s_r + 2s_\theta)^2 + (2s_r + s_\theta)^2 + 3(s_r + s_\theta)^2 \\ - 3s_r s_\theta - 9s_\theta^{2} - 6\mu(s_r + 2s_\theta) = 6k^2\end{aligned} \tag{4.1.10}$$

if the relationship

$$s_r + s_\theta + s_\phi = 0 \tag{4.1.11}$$

is used. Upon expansion of each term, equation (4.1.10) simplifies down to

$$3s_r^{\,2} - 2\mu s_r - 2k^2 = (4\mu - 3s_r)\, s_\theta .$$

Consequently,

$$s_\theta = \frac{3s_r^{\,2} - 2\mu s_r - 2k^2}{4\mu - 3s_r} \quad \text{for } s_r \neq \frac{4\mu}{3} \qquad (4.1.12)$$

and from this and equation (4.1.11),

$$s_\phi = \frac{2k^2 - 2\mu s_r}{4\mu - 3s_r} \quad \text{for } s_r \neq \frac{4\mu}{3} . \qquad (4.1.13)$$

Substituting equation (4.1.12) in equation (4.1.9) and simplifying gives

$$s_{r\theta}^{\,2} = \{- 9s_r^{\,4} + 18\mu s_r^{\,3} - 12\mu^2 s_r^{\,2} + 15k^2 s_r^{\,2} - 24\mu k^2 s_r$$

$$- 4k^4 + 16\mu^2 k^2\}\cdot\{\frac{1}{4\mu - 3s_r}\}^2 . \qquad (4.1.14)$$

The right hand side of equation (4.1.14) can be factored further to give

$$s_{r\theta}^{\,2} = \{(3s_r^{\,2} - 4k^2)(k^2 - 4\mu^2 + 6\mu s_r - 3s_r^{\,2})\}\cdot\{\frac{1}{4\mu - 3s_r}\}^2$$

and hence

$$s_{r\theta} = \frac{1}{4\mu - 3s_r}\{(4k^2 - 3s_r{}^2)(3s_r{}^2 - 6\mu s_r + 4\mu^2 - k^2)\}^{1/2} \quad (4.1.15)$$

where the positive root is taken since $s_{r\theta}$ is non-negative throughout the field. For $s_{r\theta}$ to be defined it is necessary that

$$(4 - 3\eta^2)(3\eta^2 - 6\beta\eta + 4\beta^2 - 1) \geq 0$$

where $\eta \equiv \frac{s_r}{k}$ and $\beta \equiv \frac{\mu}{k}$. The second factor $3\eta^2 - 6\beta\eta + 4\beta^2 - 1$ has discriminant $12 - 12\beta^2 < 0$ if $\beta > 1$. It is assumed that $\beta > 1$ as it appears that no solution to the flow problem is possible if $\beta \leq 1$. Consequently, the second factor has no zeros and since it has positive concavity it is positive for all η. Thus $s_{r\theta}$ is defined if $4 - 3\eta^2 \geq 0$, that is, if

$$|s_r| \leq \frac{2}{\sqrt{3}}k,$$

a condition that is clearly satisfied because of the yield criterion. From the requirement that $v_r \leq 0$, equation (4.1.2) indicates that $\frac{\partial v_r}{\partial r} \geq 0$. Then from equation (4.1.1a) together with the further requirements that $s_{r\theta} \geq 0$ and $\frac{\partial v_r}{\partial \theta} \leq 0$, one deduces that $s_r \geq 0$. Consequently, $0 \leq s_r \leq \frac{2}{\sqrt{3}}k$ and because of the assumption of axial symmetry,

$$s_r = \frac{2}{\sqrt{3}}k, \; s_\theta = s_\phi = -\frac{k}{\sqrt{3}} \text{ on } \theta = 0,$$

or for all θ if the channel is perfectly smooth, resulting in spherically symmetric flow.

From the requirement $0 \leq s_r \leq \frac{2}{\sqrt{3}} k$, it follows from equation (4.1.15) that

$$0 \leq s_{r\theta} \leq (1 - \frac{1}{4\beta^2})^{1/2} k = m'k.$$

Thus a solution cannot be obtained if the prescribed shearing stress at the surface is greater than m'k. It is only for the special case of a rigid-perfectly plastic material (m' = 1) that the prescribed shearing stress at the surface can be k.

4.2 DETERMINATION OF STRESS FIELD

Integration of the equation of equilibrium (4.1.3b) gives

$$\sigma_\theta = f(r) - \int_0^\theta \{(s_\theta - s_\phi) \cot \theta + 3s_{r\theta}\} d\theta \qquad (4.2.1)$$

since the components of the stress deviation are functions of θ only. From equation (4.2.1) an expression for the hydrostatic pressure, $- p \equiv \frac{1}{3} \sigma_k^k$, is obtained and is

$$- p = f(r) - s_\theta - \int_0^\theta \{(s_\theta - s_\phi) \cot \theta + 3s_{r\theta}\} d\theta. \qquad (4.2.2)$$

The equation of equilibrium (4.1.3a) gives

$$r \frac{\partial p}{\partial r} = - (\frac{\partial s_{r\theta}}{\partial \theta} + 3s_r + s_{r\theta} \cot\theta) \qquad (4.2.3)$$

since $\sigma_r = s_r - p$ and $2\sigma_r - \sigma_\theta - \sigma_\phi = 3s_r$. The left hand side of equation (4.2.3) is a function of r alone and the right hand side is a function of θ alone; consequently,

$$r \frac{\partial p}{\partial r} = - ck \qquad (4.2.4)$$

and

$$\frac{ds_{r\theta}}{d\theta} + 3s_r + s_{r\theta} \cot\theta = ck \qquad (4.2.5)$$

where c is constant. Since for converging flow p must increase as r decreases, c is positive.

Integration of equation (4.2.4) gives

$$p = - ck \,\ell n \left(\frac{r}{r_o}\right) - h(\theta)$$

where r_o is a constant which depends on the specified stress at some point (r,θ,ϕ) in the field and from equation (4.2.2)

$$h(\theta) = - s_\theta - \int_o^\theta \{(s_\theta - s_\phi) \cot\theta + 3s_{r\theta}\} d\theta. \qquad (4.2.6)$$

In order to obtain $s_{r\theta}$ as a function of θ, the non-linear differential equation (4.2.5) with s_r related to $s_{r\theta}$ by equation (4.1.15) must be solved subject to the conditions $s_{r\theta} = 0$ on $\theta = 0$ and $s_{r\theta} = m'k$ on $\theta = \alpha$. The constant c in equation (4.2.5) is determined by α, β and m'.

Equation (4.2.5) is written in the non-dimensional form

$$\tau'(\theta) + \tau(\theta)\cot\theta + 3\eta(\theta) = c \qquad (4.2.7)$$

where $\tau(\theta) \equiv \frac{s_{r\theta}}{k}$ and $\eta(\theta) \equiv \frac{s_r}{k}$. For the limiting case $\mu \to \infty$ for fixed k, it follows from equation (4.1.5) that $\eta(\theta) \to \frac{2}{\sqrt{3}}[1 - \tau^2(\theta)]^{1/2}$. Thus for the special case of a rigid-perfectly plastic material, equation (4.2.7) becomes

$$\tau'(\theta) + \tau(\theta)\cot\theta + 2\sqrt{3}\,[1 - \tau^2(\theta)]^{1/2} = c$$

which in non-dimensional form is the equation obtained by SHIELD for the rigid-perfectly plastic case.

To solve equation (4.2.7) a value of c was chosen and the equation was integrated numerically to give θ as a function of τ in the interval $0 \leq \tau \leq m'$ for a specified β. The solution was started at $\theta = 0$, $\tau = 0$ and continued until $\tau = m'$ was attained; the value of θ corresponding to the prescribed $\tau = m'$ at the surface being the semi-angle α of the conical channel. This required that $\frac{d\tau}{d\theta}$ be known on $\theta = 0$. From equation (4.2.7),

$$\left(\frac{d\tau}{d\theta}\right)_{\theta=0} = \lim_{\theta\to 0}\,(c - 3\eta - \tau\cot\theta)$$

$$= c - 2\sqrt{3} - \lim_{\theta\to 0}\tau\cot\theta. \qquad (4.2.8)$$

But

$$\lim_{\theta\to 0} \tau\cot\theta = \left(\frac{d\tau}{d\theta}\right)_{\theta=0}$$

and thus, from equation (4.2.8), it follows that $\left(\frac{d\tau}{d\theta}\right)_{\theta=0} = \frac{c - 2\sqrt{3}}{2}$. Since τ is non-negative in the field, $c \geq 2\sqrt{3}$ with equality holding only for spherically symmetric flow. The results of the numerical integration are shown graphically in Figure 5 for the case $\beta \equiv \frac{\mu}{k} = 10$.

The numerical solution of equation (4.2.7) involves the numerical determination of η from τ using equation (4.1.15) and substitution of the values of η so obtained in equations (4.1.12) and (4.1.13) gives the corresponding values of s_θ and s_ϕ. The results so obtained are shown graphically in Figures 6 and 7 for the cases $\beta = 10$ and $c = 10$ and 7 respectively.

The function $h(\theta)$ defined by equation (4.2.6) is also shown graphically in Figure 7 for $\beta = 10$ and $c = 10$ and 7.

4.3 DETERMINATION OF THE VELOCITY FIELD

Substituting equation (4.1.12) in equation (4.1.7) gives

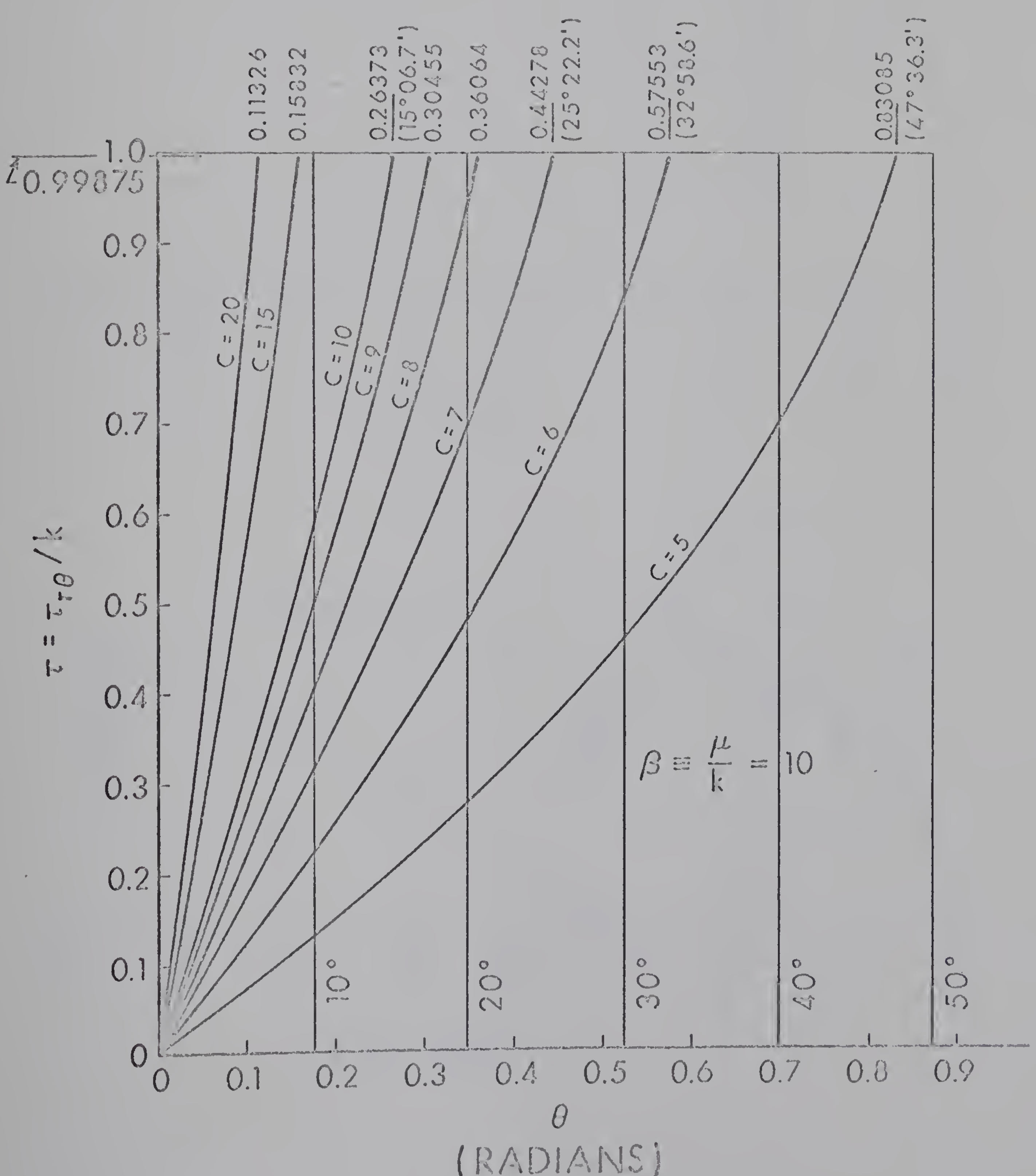
$$\frac{g'(\theta)}{g(\theta)} = \frac{2\mu}{s_{r\theta}} \left[\frac{3s_r^{\,2} - 4k^2}{2\mu s_r - 2k^2}\right]$$

$$= \frac{3\eta^2 - 4}{\tau(\eta - \frac{1}{\beta})} .$$

Hence

$$v_r = \frac{Bg(\theta)}{r^2} = \frac{B}{r^2} \exp\left[-\int_0^\theta \frac{4 - 3\eta^2}{\tau(\eta - \frac{1}{\beta})}\, d\theta\right]$$

FIGURE 5 - FUNCTION $\tau(\theta)$ FOR VARIOUS VALUES OF C



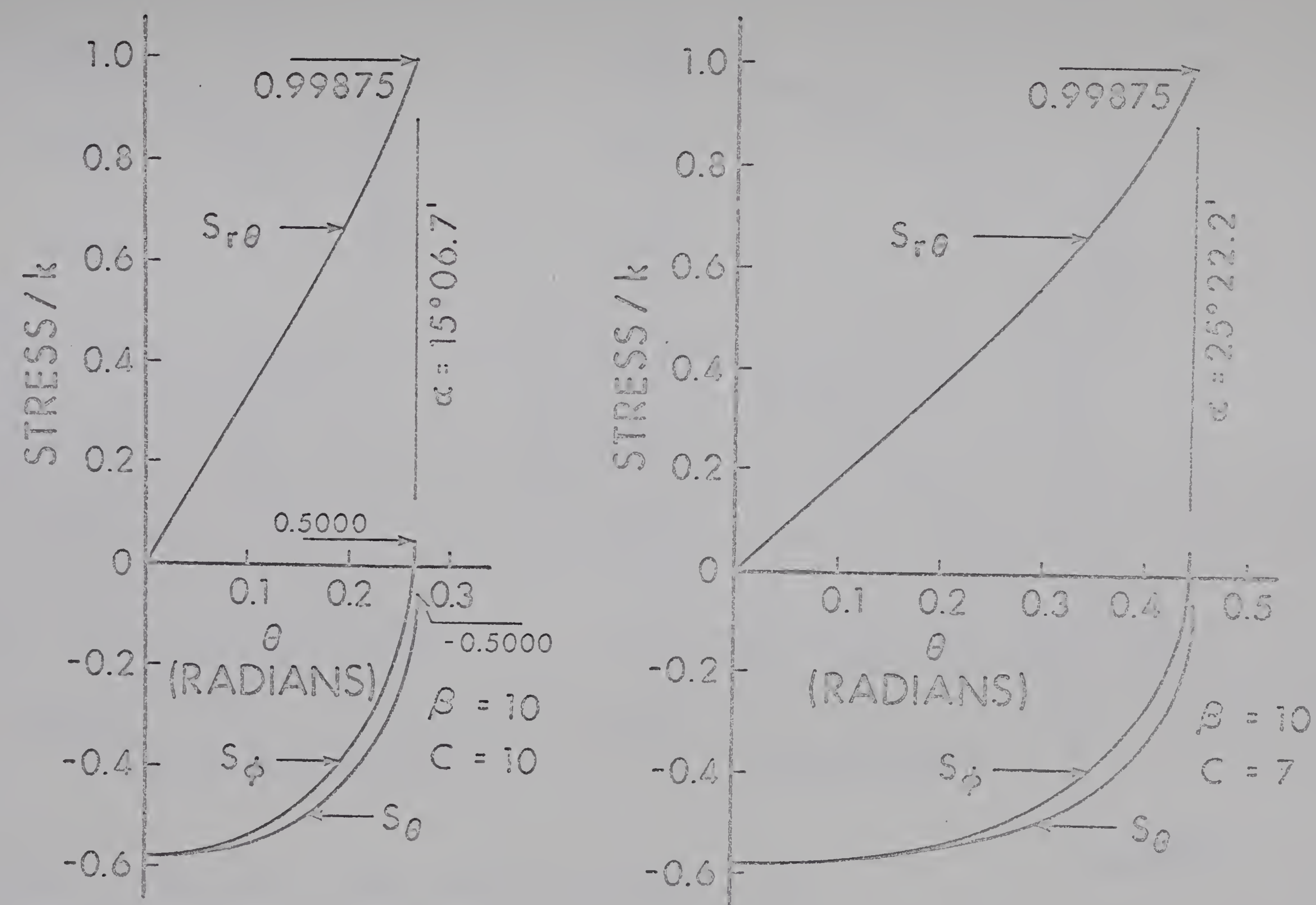


FIGURE 6 - VARIATION OF STRESS DEVIATION COMPONENTS

$s_{r\theta}$, s_ϕ, s_θ WITH θ FOR α = 15°06.7' (β = 10, C = 10) AND

α = 25°22.2' (β = 10, C = 7)

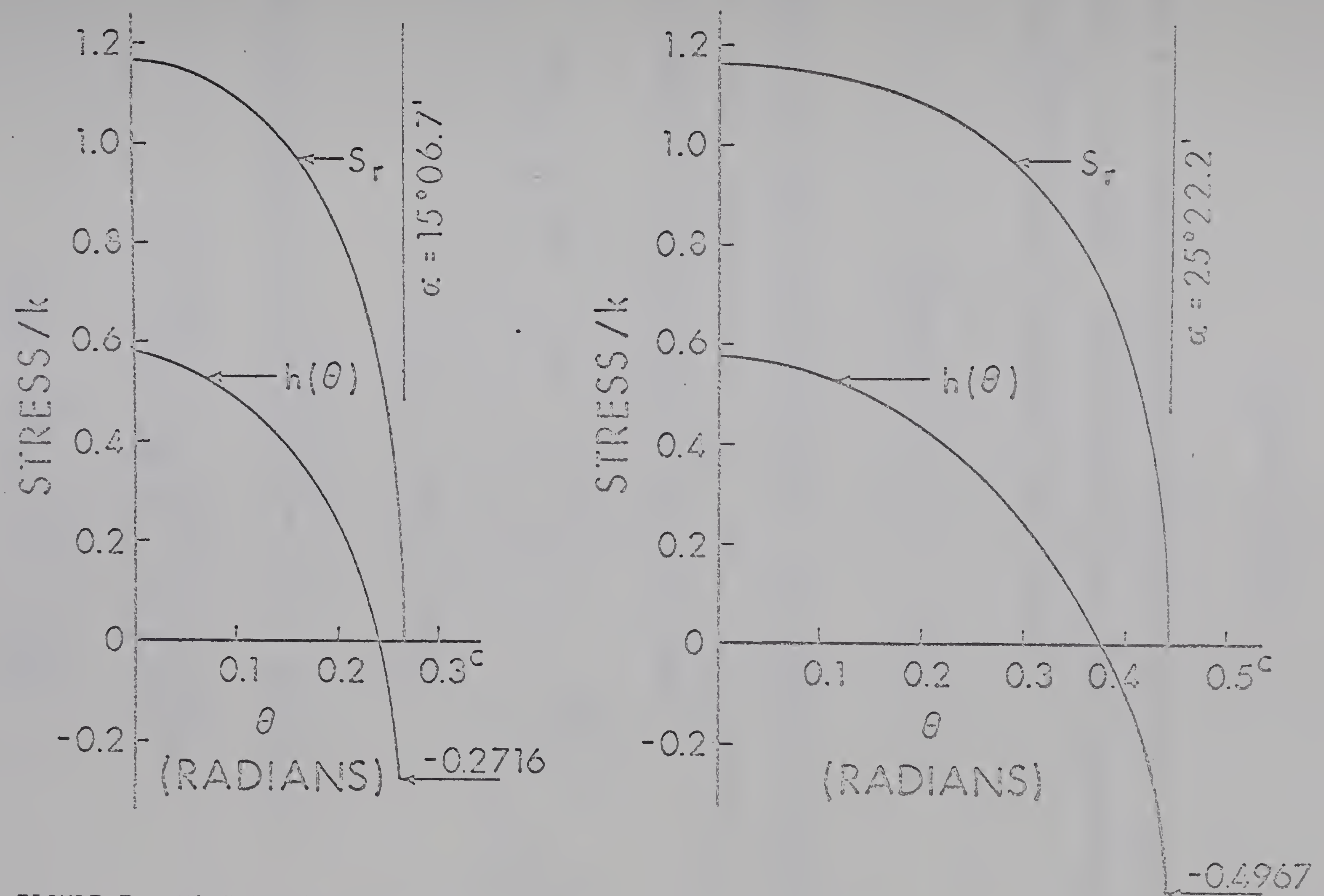


FIGURE 7 - VARIATION OF STRESS DEVIATION COMPONENTS s_r AND FUNCTION $h(\theta)$ WITH θ FOR $\alpha = 15°06.7'$ ($\beta = 10$, $C = 10$) AND $\alpha = 25°22.2'$ ($\beta = 10$, $C = 7$)

$$= \frac{B}{r^2} \exp\left[- \int_0^\theta \left\{\frac{4 - 3\eta^2}{4\beta^2 - 6\beta\eta + 3\eta^2 - 1}\right\}^{1/2} \left\{\frac{4\beta^2 - 3\beta\eta}{\beta\eta - 1}\right\} d\theta \quad (4.3.1)$$

$$= \frac{B}{r^2} \exp\left[- A(\theta)\right],$$

where B is a constant depending on the volume flow and is negative since v_r is negative. For the special case of the rigid-perfectly plastic MISES material, equation (4.3.1) becomes

$$v_r = \frac{B}{r^2} \exp\left[- 2\sqrt{3} \int_0^\theta (1 - \tau^2(\theta))^{1/2} d\theta\right]$$

which is the result obtained by SHIELD. The integrand in equation (4.3.1) has a singularity at $\eta = \frac{1}{\beta}$. For $0 \leq \theta \leq \theta_o$, where θ_o is the value of θ corresponding to $\eta = \frac{1}{\beta}$, the integral

$$A(\theta_o) \geq \frac{1}{\beta} \left[(4\beta^2 - 3)(\beta^2 - 1)\right]^{1/2} \int_0^{\theta_o} \left(- 3 + \frac{4\beta^2 - 3}{\beta} \quad \frac{1}{\eta - \frac{1}{\beta}}\right) d\theta$$

$$= \gamma(\beta)\, \theta_o + \delta(\beta) \int_0^{\theta_o} \frac{d\theta}{\eta - \frac{1}{\beta}}$$

where $\gamma(\beta)$ and $\delta(\beta)$ are positive numbers for a specified $\beta > 1$. Now

$$\int_0^{\theta_o} \frac{d\theta}{\eta - \frac{1}{\beta}} = \lim_{\varepsilon \to 0^+} \int_0^{\theta_o - \varepsilon} \frac{1}{\eta - \frac{1}{\beta}} \left(\frac{d\theta}{d\eta}\right) d\eta$$

$$\geq \lim_{\varepsilon \to 0^+} \left\{\left(\frac{d\theta}{d\eta}\right)_{\theta_o - \varepsilon} \int_0^{\theta_o - \varepsilon} \frac{d\eta}{\eta - \frac{1}{\beta}}\right\}$$

$$= \left(\frac{d\theta}{d\eta}\right)_{\theta_o} \left[\lim_{\varepsilon\to 0^+} \log \left|\eta - \frac{1}{\beta}\right|\right]_o^{\theta_o-\varepsilon} .$$

Consequently

$$\lim_{\theta\to\theta_o} \int_o^{\theta_o} \frac{d\theta}{\eta - \frac{1}{\beta}} = +\infty$$

since $\frac{d\theta}{d\eta}$ is a decreasing function and is finite and negative at $\eta = \frac{1}{\beta}$. Hence the integral $A(\theta)$ does not converge when the upper limit of integration is θ_o and $g(\theta_o) = 0$.

The scalar invariant λ, expressed in terms of the physical components of the strain rate tensor and the stress deviation tensor is

$$\lambda = \frac{1}{2k^2}\left[s_r \frac{\partial v_r}{\partial r} + \frac{s_{r\theta}}{r} \frac{\partial v_r}{\partial \theta} + (s_\theta + s_\phi) \frac{v_r}{r}\right]$$

and must be positive during plastic deformation. Using equations (4.1.5) and (4.1.7), the form of λ can be expressed as

$$\lambda = \frac{v_r}{2rk}\left[\frac{3\eta - 4\beta}{\eta\beta - 1}\right] \text{ for } v_r \neq 0 .$$

Also $3\eta - 4\beta$ is negative for $0 \leq \eta \leq \frac{2}{\sqrt{3}}$, $\beta > 1$. Thus in the region $0 \leq \theta < \theta_o$ with $v_r < 0$ and as given by equation (4.3.1), λ is positive. However in the region $\theta_o < \theta < \alpha$, where α is the value of θ corresponding to $\tau = m'$, the velocity field given by equation (4.3.1) and the stress field result in negative λ. If a positive constant B is assumed

in the region $\theta_o < \theta \leq \alpha$, then $v_r > 0$ and λ is positive, but then the positive shearing stress at $\theta = \alpha$ does not oppose the flow and does work on the material. Consequently, the velocity field is not admissible in the region $\theta_o < \theta \leq \alpha$ unless B is zero. Thus the region $\theta_o \leq \theta \leq \alpha$ does not deform during the flow. Similar non-deforming regions were indicated by the solution for the plane radial flow problem considered in CHAPTER III. Again the parameter λ has an infinite discontinuity at $\theta = \theta_o$ but this is not inconsistent with the equations (4.1.1). Of interest is the fact that this discontinuity in λ occurs when $s_r = k^2/\mu$ in both the plane converging radial flow problem and the axially symmetric converging radial flow problem just considered.

The function $g(\theta)$ in equation (4.3.1) was evaluated numerically for the cases where $\beta = 10$ and $c = 10$ and 7 respectively for the region $0 \leq \theta \leq \theta_o$ and where the maximum $s_{r\theta} = m'k$ is attained on the channel walls. Results so obtained are shown graphically in Figure 8. For practical values of $\beta \equiv \mu/k$, the angle $\alpha - \theta_o$ is very small. The relationship between $\alpha - \theta_o$ and α is shown graphically in Figure 9 for $\beta = 10$ and 100 respectively.

FIGURE 8 - GRAPHS OF $g(\theta)$ FOR $\alpha = 15°06.7'$ ($\beta = 10$, $c = 10$)

$\alpha = 25°22.2'$ ($\beta = 10$, $c = 7$)

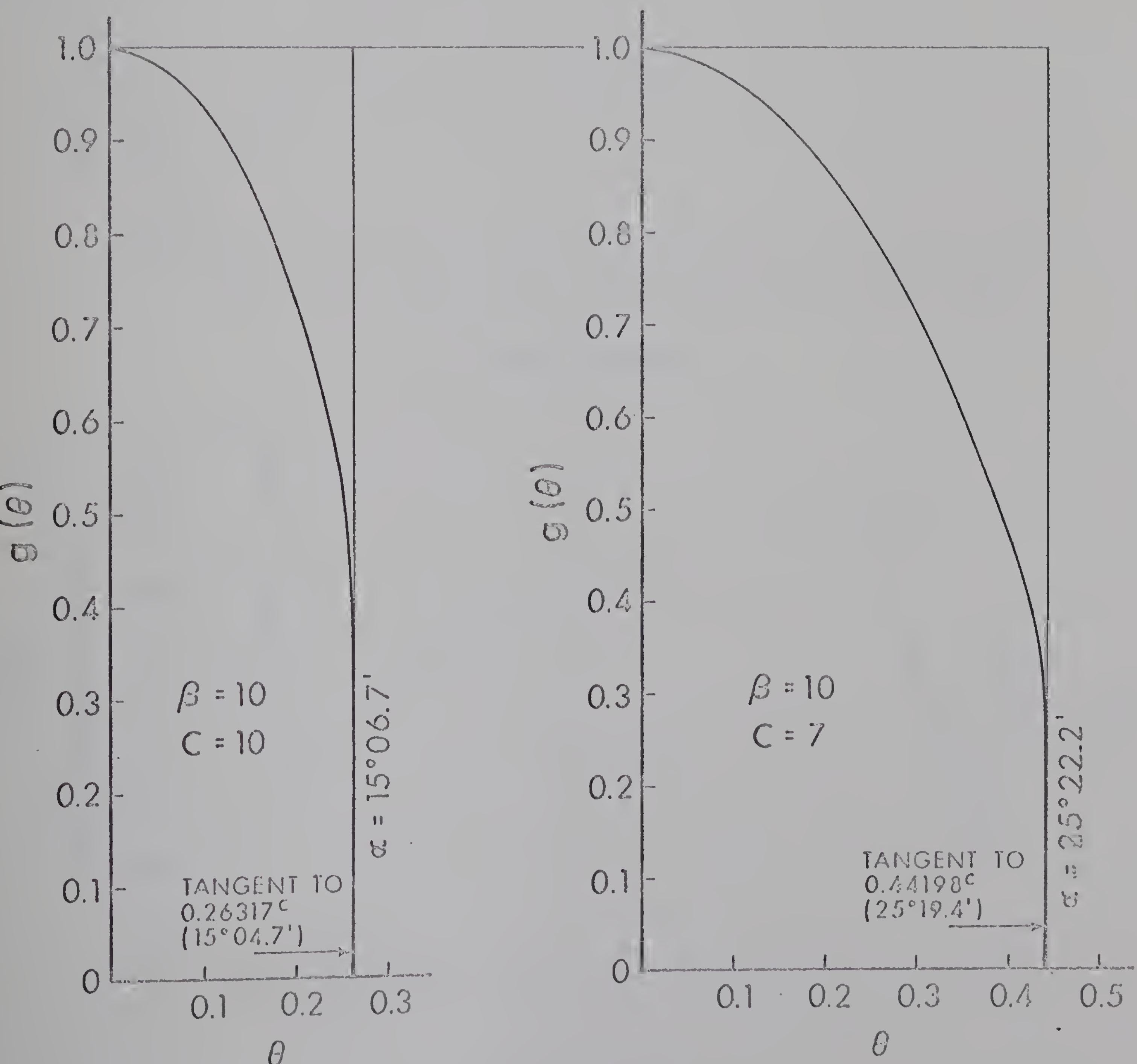
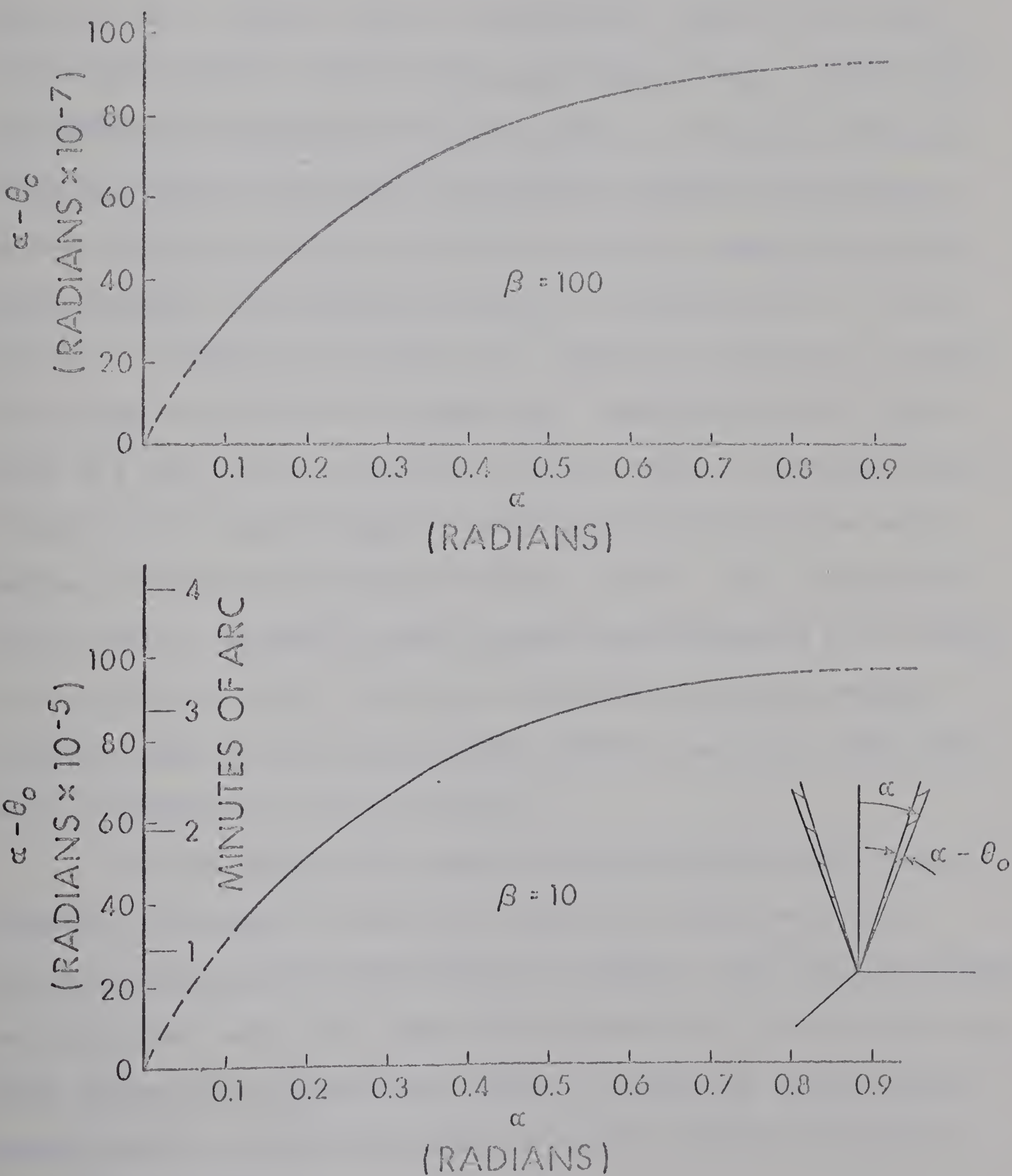


FIGURE 9 - VARIATION OF $\alpha - \theta_0$ WITH SEMI-ANGLE α

FOR β = 100 AND 10 RESPECTIVELY



CHAPTER V

PLANE RADIAL FLOW PROBLEMS OF A HYPO-ELASTIC MATERIAL OF GRADE TWO

In classical elasticity, deformations of a material are measured from a 'natural' state of the material, taken to be an equilibrium state without external forces and without stress. ZAREMBA [43] and JAUMANN [44] suggested dropping the idea of a 'natural' state and proposed, instead of the usual stress-strain relations, relationships between the rate of stress and the rate of strain. Generalizing these works, TRUESDELL [45] proposed the theory of hypo-elasticity. The researches of JAUMANN are applicable to a 'material' classified in current terminology as hypo-elastic of grade zero. TRUESDELL'S work is applicable to a wider class of materials since the theory of hypo-elasticity reduces to the classical theory of elasticity for infinitesimal deformations of the material measured from the 'natural' state configuration and may even be regarded in special cases as an enlargement of the theory of large elastic strains. Moreover, hypo-elasticity theory predicts in special cases a yield-like phenomenon called hypo-elastic yield without the assumption of a yield condition.

In this chapter, the comparison as exhibited by GREEN [46] and TRUESDELL [47] between a hypo-elastic material of grade two and an elastic-perfectly plastic MISES material is given. Also, two flow problems are considered; namely, the steady state, quasi-static, plane radial flow of an incompressible hypo-elastic material of grade two through a converging infinite channel with smooth walls and a similar problem with dynamic effects considered.

5.1 HYPO-ELASTICITY AND PLASTICITY

A hypo-elastic material is one whose constitutive equations may be written in the form

$$\frac{\delta}{\delta t}\,\sigma^{ij} = H^{ijmn}\,d_{mn} \tag{5.1.1}$$

where $H^{ijmn} = H^{jimn} = H^{ijnm}$ are the components of a hypo-elastic response function - an isotropic tensor function of the stress components σ^{ij}. TRUESDELL [48], NOLL [23], and RIVLIN and ERICKSEN [24] have shown that when the response function is a polynomial function, the components H^{ijmn} can be represented as

$$\begin{aligned}
H^{ijmn} = {} & g^{im}\,(C_1 g^{jn} + C_2\sigma^{jn} + C_3\sigma^{ir}\sigma_r{}^n) \\
& + \sigma^{im}(C_4 g^{jn} + C_5\sigma^{jn} + C_6\sigma^{jr}\sigma_r{}^n) \\
& + \sigma^{is}\sigma_s{}^m(C_1 g^{jn} + C_8\sigma^{jn} + C_9\sigma^{jr}\sigma_r{}^n) \\
& + \frac{1}{2}\,C_{10}(g^{ij}g^{mn} + g^{mj}g^{in}) \\
& + \frac{1}{2}\,C_{11}(g^{in}\sigma^{jm} + g^{mn}\sigma^{ji} + g^{jm}\sigma^{in} + g^{ji}\sigma^{mn}) \\
& + \frac{1}{2}\,C_{12}(g^{in}\sigma^{jp}\sigma_p{}^m + g^{mn}\sigma^{jp}\sigma_p{}^i + g^{jn}\sigma^{ip}\sigma_p{}^n + g^{ji}\sigma^{mp}\sigma_p{}^n),
\end{aligned}$$

where $C_1, \ldots, C_{12}$ are called the response coefficients and are polynomials in the stress invariants J_1, J_2, and J_3 defined as

$$J_1 \equiv \sigma^i_i,\ J_2 \equiv \frac{1}{2}\sigma^i_j\sigma^j_i,\ J_3 \equiv \frac{1}{3}\sigma^i_j\sigma^j_k\sigma^k_i\ .$$

Moreover, a hypo-elastic material is said to be of grade n if H^{ijmn} are polynomials of degree n in the stress components σ^{ij}. For a hypo-elastic material of grade n, $n \geq 1$, the constitutive equations (5.1.1) are form invariant under a change of objective stress rate. Not being so for n = 0, hypo-elastic materials of grade zero are considered in TRUESDELL and TOUPIN [10] as being physically irrelevant.

Since

$$\frac{\delta}{\delta t}\sigma^{ij} = \frac{\mathcal{D}\sigma^{ij}}{\mathcal{D}t} - (\sigma^{in}g^{mj} + \sigma^{nj}g^{im})d_{mn}\ ,$$

where $\frac{\mathcal{D}\sigma^{ij}}{\mathcal{D}t}$ is the JAUMANN derivative of stress given by equation (2.2.9), the constitutive equations (5.1.1) may be rewritten as

$$\frac{\mathcal{D}\sigma^{ij}}{\mathcal{D}t} = (H^{ijmn} + \sigma^{in}g^{mj} + \sigma^{nj}g^{im})\ d_{mn}$$

or, on using the mixed components σ^i_j, as

$$\frac{\mathcal{D}\sigma^i_j}{\mathcal{D}t} = g_{pj}H^{ipmn}d_{mn} + \sigma^i_p d^p_j + \sigma^p_j d^i_p\ . \qquad (5.1.2)$$

Expanding out the right hand side, equation (5.1.2) becomes

$$\frac{\mathcal{D}\sigma^i_j}{\mathcal{D}t} = C_1 d^k_k\delta^i_j + C_2\Phi\delta^i_j + C_3\Psi\delta^i_j + C_4 d^k_k\sigma^i_j$$

$$+ C_5\Phi\sigma^i_j + C_6\Psi\sigma^i_j + C_7 d^k_k\sigma^i_p\sigma^p_j + C_8\Phi\sigma^i_p\sigma^p_j$$

$$+ C_9\Psi\sigma^i_p\sigma^p_j + C_{10}d^i_j + C_{11}(d^i_p\sigma^p_j + \sigma^i_p d^p_j) \qquad (5.1.3)$$

$$+ C_{12}(d^i_k\sigma^k_p\sigma^p_j + \sigma^i_k\sigma^k_p d^p_j) + \sigma^i_p d^p_j + \sigma^p_j d^i_p,$$

where $\Phi \equiv \sigma^i_j d^j_i$ and $\Psi \equiv \sigma^i_j\sigma^j_k d^k_i$ are joint invariants of the stress tensor and the rate of strain tensor.

For the following particular choice of the response coefficients C_i ($i = 1, \ldots, 12$) as proposed by GREEN [26]:

$$C_1 = \frac{2\mu\nu}{1-2\nu} - \frac{4\mu\alpha^2}{27}(\sigma^i_i)^2, \; C_2 = C_4 = \frac{4\mu\alpha^2}{9}\sigma^i_i,$$

$$C_5 = -\frac{4\mu\alpha^2}{3}, \; C_{10} = -2\mu, \; C_{11} = -1,$$

$$C_i = 0 \text{ otherwise,}$$

and with ν the POISSON ratio and α a material constant, one obtains from equation (5.1.3) the constitutive equations for a class of compressible hypo-elastic materials of grade two, namely,

$$\frac{1}{2\mu}\frac{\mathcal{D}\sigma^i_j}{\mathcal{D}t} = d^i_j + \frac{\nu}{1-2\nu}d^k_k\delta^i_j - \frac{2\alpha^2}{3}(s^m_n f^n_m)s^i_j\,. \qquad (5.1.4)$$

Contraction of equation (5.1.4) yields

$$\frac{1}{2\mu}\frac{\mathcal{D}\sigma^k_k}{\mathcal{D}t} = \frac{1+\nu}{1-2\nu}d^k_k \qquad (5.1.5a)$$

or simply

$$\frac{Dp}{Dt} = - K\, d^k_k \; . \tag{5.1.5b}$$

Thus from equations (5.1.4) and (5.1.5a), one gets

$$\frac{1}{2\mu} \frac{\mathcal{D}s^i_j}{\mathcal{D}t} = f^i_j - \frac{2\alpha^2}{3} (s^m_n f^n_m) s^i_j$$

which together with equations (5.1.5b) are the constitutive equations of a compressible hypo-elastic material of grade two. For an incompressible hypo-elastic material of grade two, the constitutive equations are

$$\frac{1}{2\mu} \frac{\mathcal{D}s^i_j}{\mathcal{D}t} = d^i_j - \frac{2\alpha^2}{3} (s^m_n d^n_m) s^i_j$$

and (5.1.6a,b)

$$d^k_k = 0$$

which upon identifying $\frac{2\alpha^2}{3}$ with $\frac{1}{2k^2}$ are seen to be equivalent to those for the incompressible elastic-perfectly plastic MISES solid in a plastic state. For the hypo-elastic solid the second term on the right hand side of equation (5.1.6a) is present for any value of s_{ij} that occurs during deformation whereas for the elastic-perfectly plastic solid it is zero unless $s^i_j s^j_i = 2k^2$ and $s^i_j \frac{\mathcal{D}s^j_i}{\mathcal{D}t} = 0$.

By considering simple shear flow of the hypo-elastic solid with constitutive equations (5.1.6a,b), TRUESDELL [47] has shown that

as the shearing strain is increased the corresponding shearing stress first increases to a maximum value called the hypo-elastic yield and then decreases. Also, if $\frac{\mu}{k} > 1$, this shearing stress approaches, asymptotically, a value lower than the hypo-elastic yield which, along with the normal stresses required to maintain the shear flow, satisfies the VON MISES yield condition. For large values of $\frac{\mu}{k}$, which are typical for metals, the hypo-elastic yield in simple shear flow, the asymptotic value and the shearing stress predicted by the VON MISES yield condition are very nearly equal and the normal stresses required to maintain shear are negligible. The solutions to the radial flow problems considered in CHAPTER III and CHAPTER IV are not valid if $\frac{\mu}{k} < 1$ and the reason for this may be the non-existence of a VON MISES type yield for the hypo-elastic solid if $\frac{\mu}{k} < 1$. This is not of great practical importance since all common elastic-plastic solids have large values of $\frac{\mu}{k}$.

5.2 HYPO-ELASTIC FLOW IN AN INFINITE CONVERGING CHANNEL

The steady state, quasi-static plane flow of an incompressible hypo-elastic material of grade two in a converging infinite channel with smooth walls is considered. The geometry of the problem is the same as for the plane flow problem of an elastic-perfectly plastic MISES solid considered in CHAPTER III.

Since plane incompressible flow is considered, the only non-zero physical components of stress deviation and velocity are s_r, s_θ and v_r . These components are functions of r alone with $s_r = - s_\theta$ since the flow is axially symmetric. The three unknowns s_r, v_r and the hydrostatic pressure-p are determined by three governing equations; namely,

the constitutive equation

$$v_r \frac{ds_r}{dr} = 2\mu[\frac{dv_r}{dr} - \frac{2\alpha^2}{3}(\frac{dv_r}{dr} - \frac{v_r}{r})s_r^{\ 2}], \tag{5.2.1}$$

the equation of incompressibility

$$\frac{dv_r}{dr} + \frac{v_r}{r} = 0, \tag{5.2.2}$$

and the non-trivial equation of equilibrium

$$r \frac{d}{dr}(s_r - p) + 2s_r = 0, \tag{5.2.3}$$

together with an arbitrarily assigned initial value for s_r. This condition on s_r completely determines the stress field to within an arbitrary hydrostatic pressure.

Equation (5.2.2) requires that

$$v_r = -\frac{A}{r}, \ r > 0 \tag{5.2.4}$$

with the constant $A > 0$ for converging flow. Equation (5.2.1) then gives

$$\frac{ds_r}{dr} = \frac{2\mu}{kr}(s_r^{\ 2} - k^2) \tag{5.2.5}$$

where $k \equiv \frac{\sqrt{3}}{2\alpha}$, $\alpha > 0$. Integration of equation (5.2.5) yields

$$-\frac{k}{2\mu}\tanh^{-1}\frac{s_r}{k} = \log r + \log C \text{ if } |s_r| < k,$$

(5.2.6a,b,c)

$$-\frac{k}{2\mu}\coth^{-1}\frac{s_r}{k} = \log r + \log C \text{ if } |s_r| > k,$$

and

$$s_r = \pm\, k \text{ if } |s_r| = k.$$

C is a positive constant of integration. If $s_r = 0$ at $r = r_o > 0$, then from equation (5.2.6a),

$$s_r = k \tanh\left(\frac{2\mu}{k}\log\frac{r_o}{r}\right). \qquad (5.2.7)$$

Figure 10 shows the graph of $\frac{s_r}{k}$ plotted against $\frac{r}{r_o}$ and shows that s_r changes algebraic sign at $r = r_o$ and that as $\mu \to \infty$ for fixed k the curve of $\frac{s_r}{k}$ approaches pointwise the curve $\frac{s_r}{k} = 1$ in the interval $0 \leq \frac{r}{r_o} < 1$ and the curve $\frac{s_r}{k} = -1$ in the interval $1 < \frac{r}{r_o} < \infty$. If initially $s_r = k_1$, $|k_1| > k$ at $r = r_o$, then from equation (5.2.6b) it follows that s_r is infinitely discontinuous at $r = r_o$. Such behaviours in s_r are considered unrealistic and the solutions (5.2.7) and (5.2.6b) are considered inadmissible.

The initial condition $s_r = k$ at $r = r_o > 0$ gives the solution

$$s_r = k \text{ for all } r > 0, \qquad (5.2.8)$$

and from equation (5.2.3) it follows that

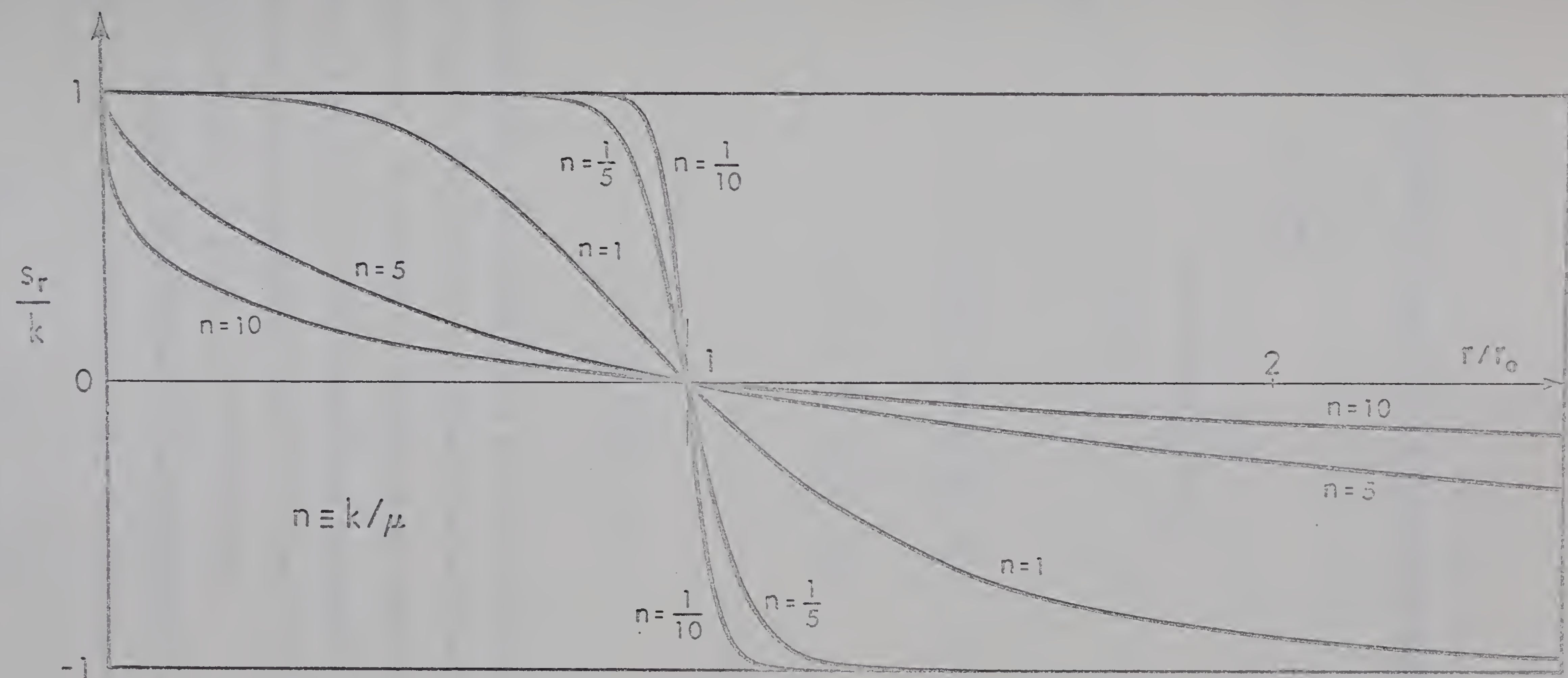


FIGURE 10 - GRAPH OF $\frac{s_r}{k}$ VERSUS $\frac{r}{r_o}$ FOR VARIOUS VALUES OF n:

$$\frac{s_r}{k} = \tanh\left(\frac{2}{n}\log\frac{r}{r_o}\right), \quad n \equiv \frac{k}{\mu}$$

$$p = 2k \log r + \log C_1,\ C_1 \text{ a positive constant.}$$

If $p = p_o$ at $r = r_o$, then

$$p = 2k \log \frac{r}{r_o} + p_o$$

and hence

$$\sigma_r = k - 2k \log \frac{r}{r_o} - p_o,$$

$$\sigma_\theta = -k - 2k \log \frac{r}{r_o} - p_o,$$

and (5.2.9a,b,c)

$$\sigma_z = - 2k \log \frac{r}{r_o} - p_o.$$

The stress field for the corresponding problem involving an elastic-perfectly plastic MISES solid is obtained from equations (3.2.1a,b and 3.1.3a) on setting $\psi = 0$. It is identical to the stress field defined by equations (5.2.9a,b,c) if in these equations the material constant k is interpreted as the asymptotic value of the shearing stress of the hypo-elastic material of grade two in simple shear.

If the above flow problem is not quasi-static and dynamic effects are taken into consideration, then the equation of motion

$$\frac{d}{dr}(s_r - p) + \frac{2s_r}{r} = \rho v_r \frac{dv_r}{dr} \qquad (5.2.10)$$

must be used rather than the equation of equilibrium (5.2.3). Since the flow is axially symmetric, the velocity v_r is a function of r only and hence from the equation of incompressibility (5.2.2),

$$v_r = -\frac{A}{r} \tag{5.2.11}$$

with the constant $A > 0$ for converging flow. The stress deviation component s_r is then as given by equations (5.2.6a,b,c) depending upon the specified initial value of s_r at $r = r_o > 0$. For the same reasons as discussed previously, the only admissible value taken for s_r is

$$s_r = +k \tag{5.2.12}$$

for all $r > 0$. Substitution of equations (5.2.11 and 5.2.12) into equation (5.2.10) and integration yields

$$p = 2k \log r - \frac{1}{2}\rho\frac{A^2}{r^2} + \log C_1, \; C_1 \text{ a positive constant.}$$

If $p = p_o$ at $r = r_o$, then

$$p = 2k \log\frac{r}{r_o} + \frac{\rho A^2}{2}\left(\frac{1}{r_o^2} - \frac{1}{r^2}\right) + p_o.$$

The stress components are then

$$\sigma_r = k - 2k \log\frac{r}{r_o} + \frac{\rho A^2}{2}\left[\frac{1}{r^2} - \frac{1}{r_o^2}\right] - p_o ,$$

$$\sigma_\theta = -k - 2k \log\frac{r}{r_o} + \frac{\rho A^2}{2}\left[\frac{1}{r^2} - \frac{1}{r_o^2}\right] - p_o ,$$

and

$$\sigma_z = - p$$

$$= - 2k \log \frac{r}{r_o} + \frac{\rho A^2}{2} \left(\frac{1}{r^2} - \frac{1}{r_o^{\,2}}\right) - p_o .$$

CHAPTER VI

CHARACTERISTIC STUDY FOR PLANE STRAIN FLOW OF AN INCOMPRESSIBLE ELASTIC-PERFECTLY PLASTIC MISES SOLID

The governing equations for the plane plastic flow of a rigid perfectly plastic MISES or TRESCA solid form a hyperbolic system and the characteristics of the stress and velocity fields coincide [49]. The characteristics, called α - lines and β - lines, are mutually orthogonal and are also called slip lines since their directions at every point coincide with those of the maximum shear strain rate. By convention, if an α - line and β - line are regarded as a pair of right-handed curvilinear reference axes, the line of action of the algebraically greatest principal stress lies in the first and third quadrant. The state of stress at a point in the deforming material is completely specified by the mean compressive stress-p and the angle ϕ which is taken to be the anti-clockwise angular rotation of the α - line from the x - axis of a CARTESIAN coordinate reference system. The two families of characteristics are defined by

$$\frac{dy}{dx} = \tan\phi \text{ for } \alpha \text{ - lines,}$$

and

$$\frac{dy}{dx} = -\cot\phi \text{ for } \beta \text{ - lines.}$$

Moreover, for the stress field, the compatibility relations are the HENCKY equations [50],

$$dp + 2k\, d\phi = 0 \text{ along an } \alpha \text{ - line,}$$

and

$$dp - 2k\, d\phi = 0 \text{ along a } \beta \text{ - line,}$$

with k the appropriate yield stress in pure shear. For the velocity field, the compatibility relations are the GEIRINGER equations [51],

$$du - v\, d\phi = 0 \text{ along an } \alpha \text{ - line,}$$

and

$$dv + u\, d\phi = 0 \text{ along a } \beta \text{ - line,}$$

where u and v are velocity components in the positive α- and β-directions respectively.

In this chapter, a corresponding study is made of the governing equations for the quasi-static, steady state plastic deformation of an incompressible elastic-perfectly plastic MISES solid in plane strain. It is shown that if $|\frac{k}{\mu}| < 1$ this system of equations admits four distinct families of real characteristics. Also compatibility relations along these characteristics are derived. In the limiting case of the rigid-perfectly plastic MISES solid ($\mu \to \infty$), the GEIRINGER equations are recovered and the stress and velocity characteristics coincide. In this chapter, cylindrical polar coordinates (r,θ,z) are used. In

APPENDIX B , the study is undertaken in a different manner and rectangular CARTESIAN coordinates are used.

For axially symmetric flow of a rigid-perfectly plastic MISES solid, PARSONS [12] has shown that the governing equations admit no real characteristics except possibly a curve in a meridional plane along which the radial velocity is zero. No corresponding study is made in this thesis of the system of governing equations for an incompressible elastic-perfectly plastic MISES solid deforming in the plastic state under axially symmetric conditions. In contrast, for a rigid-perfectly plastic TRESCA solid for which the HAAR-KÁRMÁN hypothesis has been adopted, it is known that the governing equations for stress and velocity are hyperbolic with characteristics which coincide with the slip lines in a meridional plane [52]. As yet there are no published researches pertaining to the plastic flow of an elastic-perfectly plastic TRESCA solid in finite strain.

6.1 CHARACTERISTICS OF THE STRESS FIELD AND VELOCITY FIELD FOR PLANE STRAIN DEFORMATION OF AN INCOMPRESSIBLE ELASTIC-PERFECTLY PLASTIC MISES SOLID

The governing equations, in curvilinear coordinates, for the plastic flow of an incompressible elastic-perfectly plastic MISES solid are given in CHAPTER II, SECTION 2.4 of this thesis. In APPENDIX A, SECTION A.4 these equations are expressed in cylindrical polar coordinates (r,θ,z). The number of governing equations is greatly reduced, however, for steady state, quasi-static plane flow with the deformation independent of z and parallel to the (r,θ) -plane. Since the solid is incompressible,each incremental distortion in a state of plane strain

consists only of a pure shear. The stress component σ_z normal to the plane of flow is thus equal to the mean hydrostatic pressure -p and the stress deviation component s_z vanishes identically. Also the velocity component v_z and the stress components σ_{rz} and $\sigma_{\theta z}$ are zero. It then follows from equation (A.4.3) that

$$s_r = - s_\theta \tag{6.1.1}$$

and from equation (A.4.2) that

$$s_{r\theta} = \sqrt{k^2 - s_r^2} \ . \tag{6.1.2}$$

With appropriately specified boundary conditions, the unknowns s_r, v_r, v_θ and p are determined by the constitutive equation

$$v_r \frac{\partial s_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial s_r}{\partial \theta} + \mu\left(\frac{s_r^2}{k^2} - 2\right)\frac{\partial v_r}{\partial r} + \frac{1}{r}\sqrt{k^2 - s_r^2}\left(\frac{\mu s_r}{k^2} - 1\right)\frac{\partial v_r}{\partial \theta}$$
$$+ \sqrt{k^2 - s_r^2}\left(\frac{\mu s_r}{k^2} + 1\right)\frac{\partial v_\theta}{\partial r} - \frac{\mu}{k^2}\frac{s_r}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{1}{r}\sqrt{k^2 - s_r^2}\left(\frac{\mu s_r}{k^2} + 1\right)v_\theta \tag{6.1.3a}$$
$$+ \frac{\mu s_r^2}{k^2}\frac{v_r}{r} ,$$

obtained from the PRANDTL-REUSS equation (A.4.4a) with appropriate substitutions, the equations of equilibrium

$$\frac{\partial s_r}{\partial r} - \frac{s_r}{r\sqrt{k^2 - s_r^2}}\frac{\partial s_r}{\partial \theta} - \frac{\partial p}{\partial r} = - \frac{2s_r}{r} , \tag{6.1.3b}$$

$$- \frac{s_r}{\sqrt{k^2 - s_r^2}} \frac{\partial s_r}{\partial r} - \frac{1}{r} \frac{\partial s_r}{\partial \theta} - \frac{1}{r} \frac{\partial p}{\partial \theta} = - \frac{2s_{r\theta}}{r} . \qquad (6.1.3c)$$

and the equation of incompressibility,

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = - \frac{v_r}{r} . \qquad (6.1.3d)$$

For the determination of the characteristics and compatibility relations along any existing characteristics, the above system of partial differential equations are supplemented with the relationships

$$\frac{\partial s_r}{\partial r} dr + \frac{\partial s_r}{\partial \theta} d\theta = ds_r ,$$

$$\frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta = dp, \qquad (6.1.3e,f,g,h)$$

$$\frac{\partial v_r}{\partial r} dr + \frac{\partial v_r}{\partial \theta} d\theta = dv_r ,$$

$$\frac{\partial v_\theta}{\partial r} dr + \frac{\partial v_\theta}{\partial \theta} d\theta = dv_\theta .$$

The non-zero physical components of the stress deviation tensor are parametrized in terms of the angle ψ between the line of action of the algebraically greater principal stress and a radius by setting

$$s_r = k \cos 2\psi \qquad (6.1.4)$$

and hence, by equation (6.1.2),

$$s_{r\theta} = k \sin 2\psi .$$

The characteristics of the stress field are determined from equations (6.1.3,b,c,e,f). Substituting s_r given by equation (6.1.4) in these equations:

$$\sin 2\psi \frac{\partial\psi}{\partial r} - \frac{1}{r} \cos 2\psi \frac{\partial\psi}{\partial\theta} + \frac{\partial P}{\partial r} = + \frac{1}{r} \cos 2\psi ,$$

$$\cos 2\psi \frac{\partial\psi}{\partial r} + \frac{1}{r} \sin 2\psi \frac{\partial\psi}{\partial\theta} - \frac{1}{r} \frac{\partial P}{\partial\theta} = - \frac{1}{r} \sin 2\psi , \quad (6.1.5a,b,c,d)$$

$$\frac{\partial\psi}{\partial r} dr + \frac{\partial\psi}{\partial\theta} d\theta = d\psi ,$$

and

$$\frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial\theta} d\theta = dP$$

where $P \equiv \frac{p}{2k}$. The coefficient determinant Δ_s of equations (6.1.5a,b,c,d) is

$$\Delta_s \equiv \begin{vmatrix} \sin 2\psi & -\frac{1}{r}\cos 2\psi & 1 & 0 \\ \cos 2\psi & \frac{1}{r}\sin 2\psi & 0 & -\frac{1}{r} \\ dr & d\theta & 0 & 0 \\ 0 & 0 & dr & d\theta \end{vmatrix}$$

and the determinant equation $\Delta_s = 0$ yields, after simplification, the two ordinary differential equations

$$\frac{rd\theta}{dr} = \tan\left(\psi - \frac{\pi}{4}\right)$$

and (6.1.6a,b)

$$\frac{rd\theta}{dr} = \tan\left(\psi + \frac{\pi}{4}\right).$$

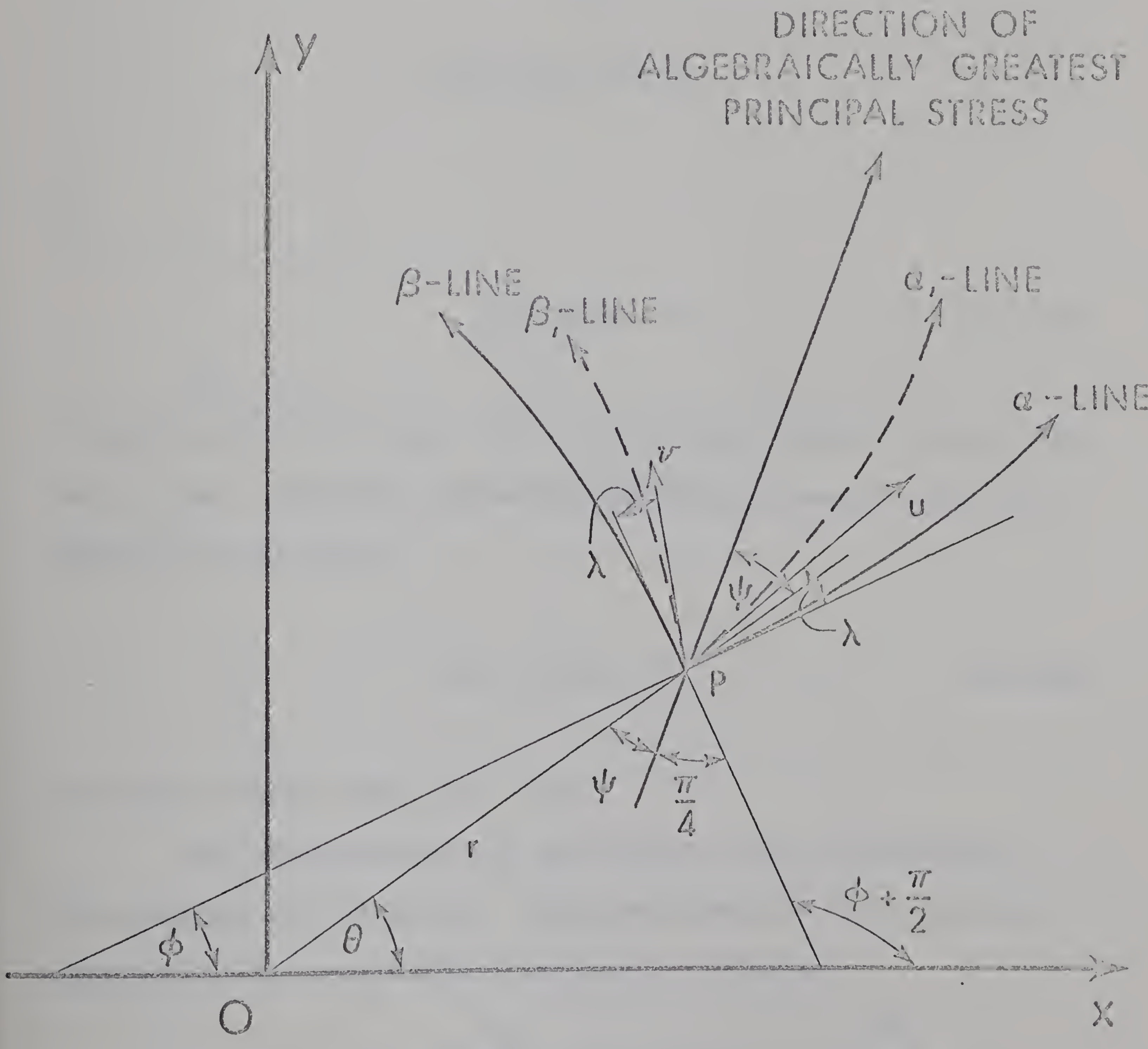
Equations (6.1.6a,b) define two families of orthogonal characteristics curves of the stress field. The system of equations (6.1.5a,b,c,d) are field equations governing the stress field and apply to any incompressible material deforming in plane strain and for which the relationship (6.1.2) is valid. Since this relationship holds for a rigid-perfectly plastic MISES or TRESCA solid in plane strain, curves defined by equations (6.1.6a,b) are also characteristics of the stress field for these solids. Figure 11 illustrates the geometrical relationship of the characteristic curves with respect to the cylindrical polar coordinate system and the CARTESIAN coordinate system. It is seen that

$$\phi + \frac{\pi}{4} = \theta + \psi \qquad (6.1.7)$$

and hence the family of characteristics defined by equation (6.1.6a) are α - lines and the family defined by equation (6.1.6b) are β - lines.

To obtain the compatibility relations along the α - lines and β - lines of the stress field, the terms in any one column of the coefficient determinant Δ_s are replaced by the corresponding terms appearing

FIGURE 11 - DIAGRAM SHOWING RELATIONSHIPS BETWEEN THE CHARACTERISTICS OF THE STRESS AND VELOCITY FIELDS AND ALSO CONVENTION USED



on the right hand side of equations (6.1.5a,b,c,d). Setting the resulting determinant equal to zero and simplifying yields

$$dP - (\sin 2\psi + \cos 2\psi \frac{dr}{rd\theta})(d\theta + d\psi) = 0. \qquad (6.1.8)$$

The compatibility relationship along an α - line is then obtained by replacing $\frac{1}{r}\frac{dr}{d\theta}$ given by equation (6.1.6a) in equation (6.1.8) to give

$$dP + d(\theta + \psi) = 0$$

or

$$dp + 2k\ d\phi = 0\ , \qquad (6.1.8a)$$

if equation (6.1.7) is used. This is the familar HENCKY relation along the α - line. Similarly, replacing $\frac{1}{r}\frac{dr}{d\theta}$ given by equation (6.1.6b) in equation (6.1.8) yields

$$dp - 2k\ d\phi = 0\ , \qquad (6.1.8b)$$

the HENCKY relation along a β - line.

The characteristics of the velocity field are determined from equations (6.1.3a,d,g,h). Using the expression for s_r given by equation (6.1.4), equation (6.1.3a) can be written as

$$(\cos^2 2\psi - 2)\frac{\partial v_r}{\partial r} + \frac{1}{r}\sin 2\psi(\cos 2\psi - \frac{k}{\mu})\frac{\partial v_r}{\partial \theta}$$

$$+ \sin 2\psi\ (\cos 2\psi + \frac{k}{\mu})\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\cos^2 2\psi\frac{\partial v_\theta}{\partial \theta}$$

$$= \frac{1}{r} \sin 2\psi \left(\cos 2\psi + \frac{k}{\mu}\right) v_\theta + \frac{1}{r} \cos^2 2\psi \, v_r$$

$$+ 2 \frac{k}{\mu} \sin 2\psi \frac{\partial \psi}{\partial r} v_r + 2 \frac{k}{\mu} \sin 2\psi \frac{\partial \psi}{\partial \theta} \frac{v_\theta}{r}. \tag{6.1.9}$$

The coefficient determinant Δ_v of equations (6.1.9) and (6.1.3d,g,h) is

$$\Delta_v \equiv \begin{vmatrix} \cos^2 2\psi - 2 & \frac{1}{r} \sin 2\psi(\cos 2\psi - \frac{k}{\mu}) & \sin 2\psi(\cos 2\psi + \frac{k}{\mu}) & -\frac{1}{r} \cos^2 2\psi \\ 1 & 0 & 0 & \frac{1}{r} \\ dr & d\theta & 0 & 0 \\ 0 & 0 & dr & d\theta \end{vmatrix}$$

which set equal to zero yields

$$r^2 \left(\frac{d\theta}{dr}\right)^2 \left(\cos 2\psi + \frac{k}{\mu}\right) - r \frac{d\theta}{dr} 2 \sin 2\psi - \left(\cos 2\psi - \frac{k}{\mu}\right) = 0 \, . \tag{6.1.10}$$

This equation is a quadratic equation in $r \frac{d\theta}{dr}$ and has discriminant $\sqrt{1 - (\frac{k}{\mu})^2}$. Thus the system of partial differential equations (6.1.9) and (6.1.3d,g,h) is hyperbolic if $|\frac{k}{\mu}| < 1$, parabolic if $|\frac{k}{\mu}| = 1$ and elliptic if $|\frac{k}{\mu}| > 1$. Considering only the case where $0 < \frac{k}{\mu} < 1$, which is the most important from the practical standpoint, and on defining

$$\frac{k}{\mu} \equiv \sin 2\gamma,$$

equation (6.1.10) yields the two differential equations:

$$r \frac{d\theta}{dr} = \tan \left(\psi + \gamma - \frac{\pi}{4}\right) \tag{6.1.11a}$$

and

$$r\,\frac{d\theta}{dr} = \tan\,(\psi - \gamma + \frac{\pi}{4}). \qquad (6.1.11b)$$

Equations (6.1.11a,b) determine two non-orthogonal families of characteristics for the velocity field. Figure 11 shows their geometrical relationship with respect to the α - and β - line of the stress field. The family of characteristics defined by equations (6.1.11a) are called α_1- lines and the family defined by equation (6.1.11b) as β_1 - lines. For the limiting case of a rigid-perfectly plastic MISES solid ($\gamma \to 0$), the characteristics of the velocity field are seen to coincide with the characteristics of the stress field.

The compatibility relationships along the α_1 - and β_1 - lines are found by firstly replacing the terms in any column of the coefficient determinant Δ_v with the corresponding terms on the right hand side of equations (6.1.9) and (6.1.3d,g,h). Setting the resulting determinant equal to zero gives

$$2\,\sin\,2\gamma\,[v_r\,\frac{\partial\psi}{\partial r} + \frac{v_\theta}{r}\,\frac{\partial\psi}{\partial\theta}] + (\cos\,2\psi + \sin\,2\gamma)$$

$$[\frac{v_\theta}{r} - \frac{dv_\theta}{ds}\,\frac{ds}{dr} - v_r\,\frac{d\theta}{dr}] - \frac{1}{r}\,(\cos\,2\psi - \sin\,2\gamma)\,\frac{dv_r}{d\theta} = 0 \quad (6.1.12)$$

where s is the arclength along a characteristic. On defining $\zeta \equiv \frac{\pi}{4} - \psi$, equation (6.1.11a) is written

$$r\,\frac{d\theta}{dr} = \tan\,(\gamma - \zeta).$$

Hence along an α_1 - line,

$$\frac{d\theta}{ds} = -\frac{1}{r}\sin(\zeta - \gamma) \tag{6.1.13a}$$

and

$$\frac{dr}{ds} = \cos(\zeta - \gamma). \tag{6.1.13b}$$

Equation (6.1.12) then becomes

$$\sin 2\gamma\left[v_r\frac{\partial\psi}{\partial r} + \frac{v_\theta}{r}\frac{\partial\psi}{\partial\theta}\right] + \frac{1}{r}\sin(\zeta+\gamma)[v_\theta\cos(\zeta-\gamma) + v_r\sin(\zeta-\gamma)] + \cos(\zeta+\gamma)\frac{dv_r}{ds} - \sin(\zeta+\gamma)\frac{dv_\theta}{ds} = 0, \tag{6.1.14}$$

if the equations (6.1.13a,b) and the trigometric relations

$$\sin 2\gamma + \cos 2\psi = 2\sin(\zeta+\gamma)\cos(\zeta-\gamma),$$

$$\sin 2\gamma - \cos 2\psi = -2\sin(\zeta+\gamma)\sin(\zeta-\gamma),$$

are used. The velocity is now resolved into the two components u and v where u is the component along the α_1 - line and v the component along the β_1 - line. Relationships between these components u and v and the components v_r and v_θ are

$$v_r = u\cos(\zeta-\gamma) + v\sin(\zeta+\gamma) \tag{6.1.15a}$$

and

$$v_\theta = - u \sin (\zeta - \gamma) + v \cos (\zeta + \gamma) \ . \qquad (6.1.15b)$$

With the use of these relationships, it follows that

$$\frac{1}{r} \sin (\zeta + \gamma)[v_\theta \cos (\zeta - \gamma) + v_r \sin (\zeta - \gamma)]$$

$$= \frac{v}{r} \sin (\zeta + \gamma) \cos 2\gamma \qquad (6.1.16)$$

and

$$\cos (\zeta + \gamma) \frac{dv_r}{ds} - \sin (\zeta + \gamma) \frac{dv_\theta}{ds} = \cos 2\gamma \frac{du}{ds}$$

$$+ (u \sin 2\gamma + v) \frac{d\zeta}{ds} \ . \qquad (6.1.17)$$

Now an expression for $\frac{d\zeta}{ds}$ along an α_1 - line is presently derived involving the variable $P \equiv \frac{p}{2k}$. This leads to a more concise form for the compatibility relation along an α_1 - line. Since $\zeta = \pi/4 - \psi$, then

$$\frac{\partial\psi}{\partial r} = - \frac{\partial\zeta}{\partial r} \ ,$$

$$\qquad (6.1.18a,b)$$

$$\frac{\partial\psi}{\partial\theta} = - \frac{\partial\zeta}{\partial\theta} \ ,$$

and the equations (6.1.5a,b) become

$$\cos 2\zeta \frac{\partial\zeta}{\partial r} - \frac{1}{r} \sin 2\zeta \frac{\partial\zeta}{\partial\theta} = \frac{\partial P}{\partial r} - \frac{1}{r} \sin 2\zeta,$$

and

$$\sin 2\zeta \frac{\partial\zeta}{\partial r} + \frac{1}{r} \cos 2\zeta \frac{\partial\zeta}{\partial\theta} = -\frac{1}{r}\frac{\partial P}{\partial\theta} + \frac{1}{r} \cos 2\zeta.$$

Solving for $\frac{\partial\zeta}{\partial r}$ and $\frac{\partial\zeta}{\partial\theta}$ gives

$$\frac{\partial\zeta}{\partial r} = \cos 2\zeta \frac{\partial P}{\partial r} - \frac{1}{r} \sin 2\zeta \frac{\partial P}{\partial\theta} ,$$

and (6.1.19a,b)

$$\frac{\partial\zeta}{\partial\theta} = 1 - (r \sin 2\zeta \frac{\partial P}{\partial r} + \cos 2\zeta \frac{\partial P}{\partial\theta}) .$$

Hence along an α_1 - line,

$$\frac{d\zeta}{ds} = \cos(\zeta + \gamma) \frac{\partial P}{\partial r} - \frac{1}{r} \sin(\zeta + \gamma) \frac{\partial P}{\partial\theta} - \frac{1}{r} \sin(\zeta - \gamma).$$

Equation (6.1.17) then becomes

$$\cos(\zeta + \gamma) \frac{dv_r}{ds} - \sin(\zeta + \gamma) \frac{dv_\theta}{ds} = \cos 2\gamma \frac{du}{ds} - (u \sin 2\zeta + v) \tag{6.1.20}$$

$$\frac{1}{r} \sin(\zeta - \gamma) + (u \sin 2\gamma + v)[\cos(\zeta + \gamma) \frac{\partial P}{\partial r} - \frac{1}{r} \sin(\zeta + \gamma) \frac{\partial P}{\partial\theta}].$$

Also from equations (6.1.15a,b) and (6.1.19a,b),

$$\sin 2\gamma[v_r \frac{\partial\psi}{\partial r} + \frac{v_\theta}{r} \frac{\partial\psi}{\partial\theta}] = \sin 2\gamma[u\{-\cos(\zeta + \gamma) \frac{\partial P}{\partial r} + \sin(\zeta + \gamma) \frac{1}{r} \frac{\partial P}{\partial\theta}\}$$

$$+ v\{\sin(\zeta - \gamma) \frac{\partial P}{\partial r} + \cos(\zeta - \gamma) \frac{1}{r} \frac{\partial P}{\partial\theta}\}$$

$$+ \frac{1}{r} \{u \sin(\zeta - \gamma) - v \cos(\zeta - \gamma)\}]. \tag{6.1.21}$$

Substitution of equations (6.1.20) and (6.1.21) into equation (6.1.14), after simplification, results in

$$\cos 2\gamma \, [\frac{du}{ds} + v \, (\frac{\partial P}{\partial r} \cos \, (\zeta - \gamma) - \frac{\partial P}{\partial \theta} \frac{1}{r} \sin \, (\zeta - \gamma)] = 0$$

or, with use of equations (6.1.13a,b),

$$\cos 2\gamma \, [\frac{du}{ds} + v(\frac{\partial P}{\partial r} \frac{dr}{ds} + \frac{\partial P}{\partial \theta} \frac{d\theta}{ds})] = 0. \tag{6.1.22}$$

Thus the compatability relation along the α_1 - line of the velocity field is

$$\frac{du}{ds} + \frac{v}{2k} \frac{dp}{ds} = 0$$

or

$$du + \frac{v}{2k} \, dp = 0 \, . \tag{6.1.23}$$

As observed earlier for the limiting case for the rigid-plastic MISES solid ($\gamma \to 0$), the characteristics of the velocity field coincide with the characteristics of the stress field. Along the α - lines, the HENCKY relation (6.1.18a) holds and hence, in the limiting case $\gamma \to 0$, the relation (6.1.23) becomes

$$du - v d\phi = 0,$$

the GEIRINGER equation for an α - line of the velocity field of a deforming rigid-perfectly plastic MISES solid.

The compatibility relation along a β_1 - line may be obtained in an exactly analogous manner to that given above for the α_1 - line. It follows from equation (6.1.11b), which must be used rather than equation (6.1.11a), that

$$\frac{d\theta}{ds} = \frac{1}{r}\cos(\zeta + \gamma),$$
$$\frac{dr}{ds} = \sin(\zeta + \gamma)\ , \qquad (6.1.24)$$

with s the arclength along a β_1 - characteristic. Hence from equations (6.1.5a,b), it follows that

$$\frac{d\zeta}{ds} = \frac{1}{r}\cos(\zeta + \gamma) - \sin(\zeta - \gamma)\frac{\partial P}{\partial r} - \frac{1}{r}\cos(\zeta - \gamma)\frac{\partial P}{\partial \theta}\ . \qquad (6.1.25)$$

Substitution of equations (6.1.15a,b), (6.1.24) and (6.1.25) into equation (6.1.14) yields, after simplification, the relation

$$\cos 2\gamma\left(\frac{dv}{ds} + u\frac{dP}{ds}\right) = 0\ ,$$

or along a β_1 - line.

$$dv + \frac{u}{2k}dp = 0. \qquad (6.1.26)$$

For the limiting case $\gamma \to \infty$, equation (6.1.26) becomes

$$dv + u d\phi = 0 \ ,$$

the GEIRINGER equation for the β_1 - line of the velocity field of a rigid perfectly plastic MISES solid.

Compatibility relations (6.1.23) and (6.1.26) are observed to have exactly the same form as the GEIRINGER equations if the latter equations are written in terms of u, v and p rather than u, v and θ as is customary in plasticity literature.

6.2 DETERMINATION OF THE CHARACTERISTICS OF THE STRESS AND VELOCITY FIELDS FOR THE CONVERGING PLANE FLOW PROBLEM

In this section, the characteristics of the stress and velocity fields are determined for the plane strain flow problem considered in CHAPTER III and which involved a converging infinite channel with perfectly rough sides. Since the stress and velocity fields have already been determined, no use is made of the method of characteristics in determining these fields.

In SECTION 6.1, the differential equations for the characteristics of the stress field for the plastic flow of an elastic-perfectly plastic MISES solid in plane strain are given by equations (6.1.6a,b). For any specific problem, the differential equations can be solved if either ψ is a known function of r and θ or $\frac{d\theta}{d\psi}$ is a known function of ψ. For the problem considered in CHAPTER III, $\frac{d\theta}{d\psi}$ is given by equation (3.2.3b). Hence equations (6.1.6a,b) may be written as

$$\frac{dr}{r} = - \left(\frac{1 + \sin 2\psi}{c - \cos 2\psi}\right) d\psi \qquad \text{for an } \alpha \text{ - line,} \quad (6.2.1a)$$

and

$$\frac{dr}{r} = \frac{1 - \sin 2\psi}{c - \cos 2\psi}\, d\psi \quad \text{for a } \beta \text{ - line} \qquad (6.2.1b)$$

where

$$c > 1.1922.$$

The constant c appearing in equations (6.2.1a,b) is, of course, a function of the semi-angle α of the converging infinite channel as given by equation (3.2.6). With a change of variable from ψ to $z = \tan\psi$, equations (6.2.1a,b) become, respectively,

$$\frac{dr}{r} = [\frac{z}{z^2+1} + \frac{1}{c+1} \cdot \frac{1}{z^2 + (\frac{c-1}{c+1})} - \frac{z}{z^2 + (\frac{c-1}{c+1})}]\, dz \text{ for an } \alpha \text{ - line },$$

and

$$\frac{dr}{r} = [\frac{z}{z^2+1} - \frac{1}{c+1} \cdot \frac{1}{z^2 + (\frac{c-1}{c+1})} - \frac{z}{z^2 + (\frac{c-1}{c+1})}]\, dz \text{ for a } \beta \text{ - line},$$

where the right hand side of each equation is expressed in partial fraction form. Integration of these two differential equations give the following equations for the characteristic curves:

$$r = b\sqrt{\frac{c+1}{c-\cos 2\psi}}\, \exp\, [\frac{1}{\sqrt{c^2-1}} \tan^{-1} (\sqrt{\frac{c+1}{c-1}} \tan\psi)] \text{ for an } \alpha \text{ - line}, \qquad (6.2.2a)$$

and

$$r = b\sqrt{\frac{c+1}{c-\cos 2\psi}}\exp\left[-\frac{1}{\sqrt{c^2-1}}\tan^{-1}\left(\sqrt{\frac{c+1}{c-1}}\tan\psi\right)\right] \text{ for a } \beta \text{ - line,} \tag{6.2.2b}$$

The constant b is the constant of integration and the angle ψ varies between $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. If r_0 is the value of r at which an α - line or β - line intersects the axis $\theta = \psi = 0$, then $b = r_0\sqrt{\frac{c-1}{c+1}}$.

The equations (6.2.2a,b) are also the equations for the slip lines of stress field. NADAI [40] obtained equations (6.2.2a,b) as the equations of the slip lines in the flow problem of a rigid plastic solid in a perfectly rough converging channel and plotted curves of the slip lines for the cases where $\alpha = 24°17'$ (c=2) and $\alpha = 90°$ (c=1.1922). It follows from equations (6.2.2a,b) that for $r_0 > 0$ the characteristics of the stress field terminate on the channel walls at finite distances from the virtual apex of the converging channel.

The characteristics of the velocity field for the plastic flow problem of an elastic-perfectly plastic MISES solid in plane strain and for which $0 < \frac{k}{\mu} < 1$ are determined from equations (6.1.11a,b). For the flow problem under consideration, these differential equations, with the use of equation (3.2.3b), may be expressed as

$$\frac{dr}{r} = \frac{\sqrt{1-n^2} + \sin 2\psi}{n - \cos 2\psi}\,\frac{\cos 2\psi}{c - \cos 2\psi}\,d\psi \qquad \text{for an } \alpha_1 \text{ - line,}$$

and (6.2.3a,b)

$$\frac{dr}{r} = \frac{\sqrt{1-n^2} - \sin 2\psi}{\cos 2\psi - n}\,\frac{\cos 2\psi}{c - \cos 2\psi}\,d\psi \qquad \text{for an } \beta_1 \text{ - line.}$$

Again c > 1.1922 and is related to α by equation (3.2.5). Also $n \equiv \frac{k}{\mu}$. With a change of variable from ψ to $z = \tan\psi$, equations (6.2.3a,b) become

$$\frac{dr}{r} = \frac{1}{c+1}\sqrt{\frac{1-n}{1+n}}\left[\frac{\left(1 - \frac{2z}{\sqrt{1-n^2}} + z^2\right)(1 - z^2)}{\left(\frac{1-n}{1+n} - z^2\right)\left(\frac{c-1}{c+1} + z^2\right)(1 + z^2)}\right] dz \text{ for an } \alpha_1 \text{ - line,}$$

and

$$\frac{dr}{r} = \frac{1}{c+1}\sqrt{\frac{1-n}{1+n}}\left[\frac{\left(1 + \frac{2z}{\sqrt{1-n^2}} + z^2\right)(1 - z^2)}{\left(z^2 - \frac{1-n}{1+n}\right)\left(\frac{c-1}{c+1} + z^2\right)(1 + z^2)}\right] dz \text{ for a } \beta_1 \text{ - line.}$$

Integration of these equations give the following characteristic curves of the velocity field valid provided $\cos 2\psi \neq n$:

$$r = b_1 \frac{\left(\left|\tan\psi + \sqrt{\frac{1-n}{1+n}}\right|\right)^{\frac{n}{c-n}}}{\left(\frac{c-1}{c+1} + \tan^2\psi\right)^{\frac{c}{2(c-n)}}\cos\psi} \exp\left[\frac{c}{c-n}\sqrt{\frac{1-n^2}{c^2-1}}\tan^{-1}\left(\sqrt{\frac{c+1}{c-1}}\tan\psi\right)\right] \quad \text{for } \alpha_1 \text{ - lines,}$$

and (6.2.4a,b)

$$r = b_1 \frac{\left(\left|\tan\psi - \sqrt{\frac{1-n}{1+n}}\right|\right)^{\frac{n}{c-n}}}{\left(\frac{c-1}{c+1} + \tan^2\psi\right)^{\frac{c}{2(c-n)}}\cos\psi} \exp\left[-\frac{c}{c-n}\sqrt{\frac{1-n^2}{c^2-1}}\tan^{-1}\left(\sqrt{\frac{c+1}{c-1}}\tan\psi\right)\right] \quad \text{for } \beta_1 \text{ - lines.}$$

The constant of integration b_1 is related to r_0, the value of r at which the α_1 - or β_1 - line intersects the axis $\theta = \psi = 0$, by

$$b_1 = r_0 \left[\left(\frac{c-1}{c+1}\right)^c \left(\frac{1-n}{1+n}\right)^{-n}\right]^{\frac{1}{2(c-n)}} .$$

For $n \to 0$, it is noted that equations (6.2.4a,b), in the limit, become equations (6.2.2a,b). This follows since

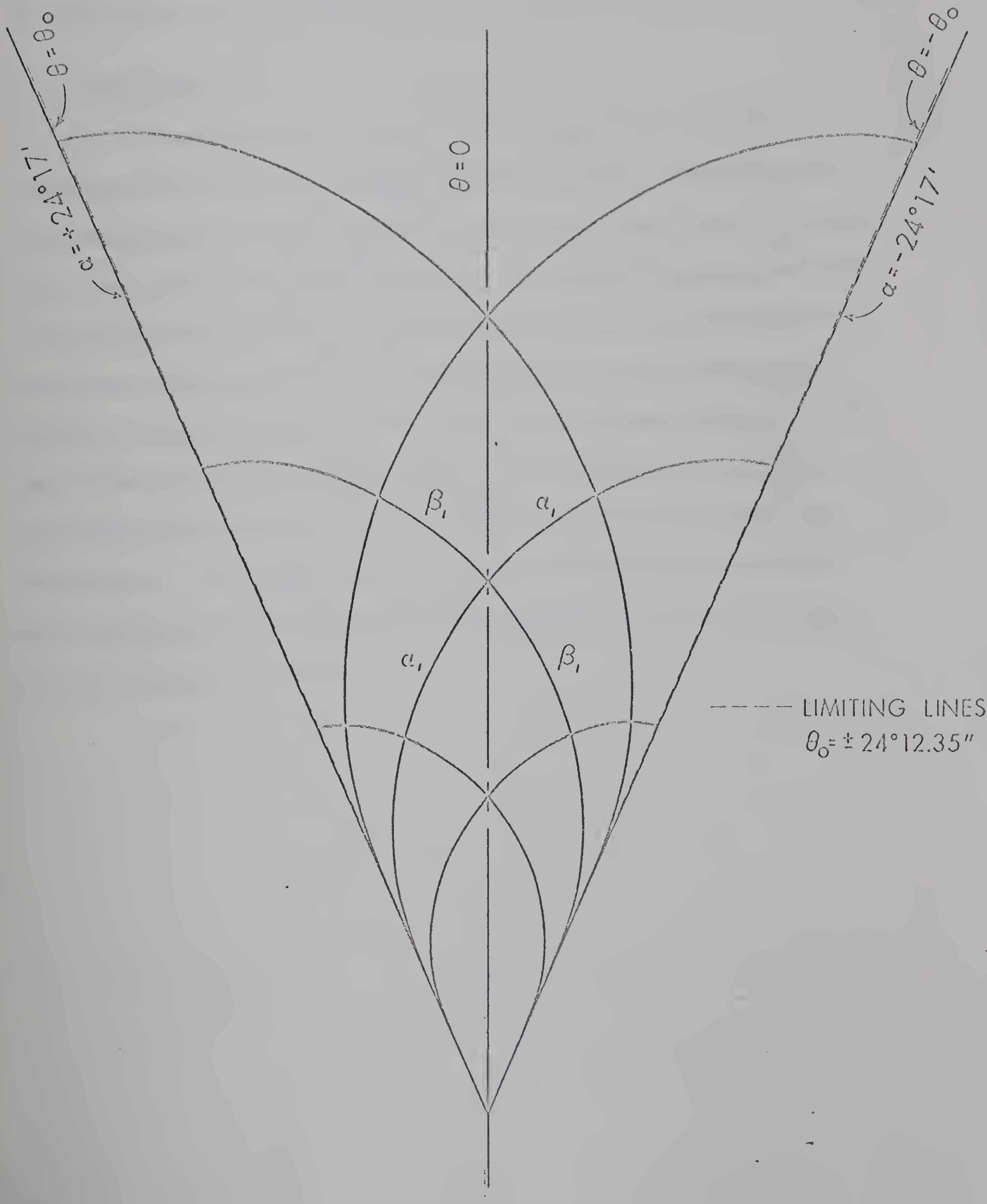
$$\frac{1}{(\cos\psi)\left(\frac{c-1}{c+1} + \tan^2\psi\right)^{1/2}} = \sqrt{\frac{c+1}{c-\cos 2\psi}}$$

and $\lim\limits_{n\to 0} b_1 = b$. This result is expected, since for plane strain flow of a rigid perfectly plastic MISES solid, the characteristics of stress and velocity coincide.

Figure 12 shows the characteristics of the velocity field of the flow problem considered in CHAPTER III for an elastic-perfectly plastic MISES with $\frac{k}{\mu} = 0.10$ and a perfectly rough channel with semi-angle $\alpha = 24°17'(c=2)$. In the region $-\theta_0 \leq \theta \leq \alpha$, corresponding to $-\frac{1}{2}\cos^{-1}\frac{k}{\mu} \leq \psi \leq \frac{\pi}{4}$, the α_1 - lines are tangent to the radial line $\theta = -\theta_0$ at the virtual apex $r = 0$ and terminate on the radial line $\theta = \alpha$. In the region $-\alpha \leq \theta \leq -\theta_0$, corresponding to $-\frac{\pi}{4} \leq \psi \leq -\frac{1}{2}\cos^{-1}\frac{k}{\mu}$, the α_1 - lines are again tangent to $\theta = -\theta_0$ at $r = 0$ but terminate on $\theta = -\alpha$. The β_1 - lines are reflections of α_1 - lines about the axis $\theta = 0$. The radial lines $\theta = \pm\theta_0$ are limiting lines of the velocity characteristic field. Since $\pm\theta_0$ are values of θ corresponding respectively to values of ψ such that as $\cos 2\psi = \frac{k}{\mu}$, these limiting lines coincide with the inner boundaries of the non-deforming

FIGURE 12 - CHARACTERISTICS OF THE VELOCITY FIELD FOR CONVERGING FLOW IN A PERFECTLY ROUGH CHANNEL

$\alpha = 24°17'$ $(c = 2)$, $n \equiv \frac{k}{\mu} = 0.10$



regions exhibited in CHAPTER III. If the frictional shearing stress on the channel walls is equal to $\sqrt{1 - n^2}\, k$, $n \equiv \frac{k}{\mu}$, then the limiting lines coincide with the channel walls; if less then they fall outside the channel walls and the end points of the characteristic curves lie on the channel walls at finite distances from the virtual apex.

6.3 CONCLUSIONS

The characteristic study in this chapter has involved a frame invariant form of the PRANDTL-REUSS equations. Provided that the incompressible elastic-perfectly plastic MISES solid has a $\frac{k}{\mu}$ value less than unity, the governing equations for plane flow are hyperbolic and hence the method of characteristics can be used for solutions of problems, either rotational or irrotational, involving finite deformations. Extensive search of the literature indicates that much of the researches concerning the application of characteristics to elasto-plasticity has been restricted to problems involving only small deformations. It is hoped that the results developed here can be used both in extending our knowledge of elasto-plasticity and in solving many technological problems.

APPENDIX A

GOVERNING EQUATIONS IN A SPHERICAL POLAR AND A CYLINDRICAL POLAR COORDINATE SYSTEM FOR AN ELASTIC-PERFECTLY PLASTIC MISES SOLID IN FINITE STRAIN

In CHAPTER II, SECTION 2.4 of this thesis, the governing equations for an elastic-perfectly plastic MISES solid in finite strain are given with respect to a general curvilinear coordinate system. In this APPENDIX A, these governing equations are expressed in both a spherical polar and a cylindrical polar coordinate system using the physical components of the vector and tensor quantities involved. The detailed transformation of the PRANDTL-REUSS constitutive equations is included. However, since the field equations expressed in these two coordinate systems are contained in various texts, notably McCONNELL [53] and LANGLOIS [54], these equations are only entered here for easy reference.

A.1 CHRISTOFFEL SYMBOLS AND PHYSICAL COMPONENTS FOR A SPHERICAL POLAR COORDINATE SYSTEM

For a spherical polar coordinate system r, θ, ϕ as illustrated in Figure 4, the line element is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 .$$

The covariant and contravariant components of the metric tensor are

$$g_{11} = 1,\ g_{22} = r^2,\ g_{33} = r^2\sin^2\theta,\ g_{ij} = 0(i\neq j) ,$$

and

$$g^{11} = 1,\ g^{22} = \frac{1}{r^2},\ g^{33} = \frac{1}{r^2\sin^2\theta},\ g^{ij} = 0\ (i \neq j);$$

and the non-zero CHRISTOFFEL symbols of the second kind are

$$\Gamma^1_{22} = -\ r,\ \Gamma^1_{33} = -\ r\sin^2\theta,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \Gamma^2_{33} = -\ \sin\theta\cos\theta,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\ \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

In terms of the corresponding covariant and contravariant components, the physical components of velocity are

$$v_r = v_1 = v^1,\ v_\theta = \frac{v_2}{r} = rv^2,\ v_\phi = \frac{v_3}{r\sin\theta} = r\sin\theta v^3,$$

and the physical components of the symmetric stress tensor are

$$\sigma_r = \sigma_{11} = \sigma^{11},\ \sigma_\theta = \frac{\sigma_{22}}{r^2} = r^2\sigma^{22},\ \sigma_\phi = \frac{\sigma_{33}}{r^2\sin^2\theta} = r^2\sin^2\theta\sigma^{33},$$

$$\tau_{r\theta} = \frac{\sigma_{12}}{r} = r\sigma^{12},\ \tau_{r\phi} = \frac{\sigma_{13}}{r\sin\theta} = r\sin\theta\sigma^{13},\ \tau_{\theta\phi} = \frac{\sigma_{23}}{r^2\sin\theta} = r^2\sin\theta\sigma^{23}.$$

The physical components of the stress deviation tensor, s_r, s_θ, s_ϕ, $s_{r\theta}$, $s_{r\phi}$ and $s_{\theta\phi}$, are similarly related to the covariant and contravariant components of the stress deviation tensor. From the relationships

$$s^{ij} = \sigma^{ij} + pg^{ij}, \ p = -\frac{1}{3} g_{mn} \sigma^{mn} ,$$

it follows that

$$s_r = \sigma_r + p, \ s_\theta = \sigma_\theta + p, \ s_\phi = \sigma_\phi + p,$$

$$s_{r\theta} = \tau_{r\theta}, \ s_{r\phi} = \tau_{r\phi}, \ s_{\theta\phi} = \tau_{\theta\phi},$$

with

$$p = -\frac{1}{3} (\sigma_r + \sigma_\theta + \sigma_\phi).$$

The physical components of the symmetric strain rate tensor are

$$d_r = \frac{\partial v_r}{\partial r} , \ d_\theta = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} , \ d_\phi = \frac{v_r}{r} + \frac{\cot\theta}{r} v_\theta + \frac{1}{r\sin\theta} \frac{\partial v_\phi}{\partial \phi} ,$$

$$d_{r\theta} = \frac{1}{2}\left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right), \ d_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta} \frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right),$$

$$d_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta} \frac{\partial v_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{\cot\theta}{r} v_\phi\right),$$

and the physical components of the skew-symmetric vorticity tensor are

$$\omega_{r\theta} = \frac{1}{2r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (rv_\theta)\right], \ \omega_{r\phi} = \frac{1}{2}\left[\frac{1}{r} \frac{\partial}{\partial r}(rv_\phi) - \frac{1}{r\sin\theta} \frac{\partial v_r}{\partial \phi}\right],$$

$$\omega_{\theta\phi} = \frac{1}{2r\sin\theta} \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta} (v_\phi \sin\theta)\right].$$

A.2 GOVERNING EQUATIONS IN SPHERICAL POLAR COORDINATES

The equations of motion (2.4.2) in terms of the physical components of stress and velocity are

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} + \frac{1}{r}(2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{r\theta}\cot\theta) = \rho(\frac{Dv_r}{Dt} - F_r),$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{1}{r}[3\tau_{r\theta} + (\sigma_\theta - \sigma_\phi)\cot\theta] = \rho(\frac{Dv_\theta}{Dt} - F_\theta), \quad \text{(A.2.1a,b,c)}$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_\phi}{\partial\phi} + \frac{1}{r}(3\tau_{r\phi} + 2\tau_{\theta\phi}\cot\theta) = \rho(\frac{Dv_\phi}{Dt} - F_\phi),$$

where

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}(v_\theta{}^2 + v_\phi{}^2),$$

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + \frac{v_r}{r}\frac{\partial}{\partial r}(rv_\theta) + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{\cot\theta}{r}v_\phi{}^2,$$

$$\frac{Dv_\phi}{Dt} = \frac{\partial v_\phi}{\partial t} + \frac{v_r}{r}\frac{\partial}{\partial r}(rv_\phi) + \frac{v_\theta}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\phi) + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi},$$

ρ is the density and F_r, F_θ, F_ϕ are the physical components of the body force per unit mass. For quasi-static problems in the absence of body forces, the right hand sides of equations (A.2.1a,b,c) are zero. These equations are then called the equations of equilibrium.

The equation of continuity (2.4.3) in terms of the density and the physical components of velocity is

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\rho v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v_\phi) = 0.$$

If the solid is incompressible, there is also the equation of incompressibility (2.4.1b),

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\cot\theta}{r} v_\theta + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0.$$

The identity (2.4.5) is

$$s_r + s_\theta + s_\phi = 0, \qquad \text{(A.2.2)}$$

and the VON MISES yield condition (2.4.4) is

$$s_r{}^2 + s_\theta{}^2 + s_\phi{}^2 + 2(s_{r\theta}{}^2 + s_{\theta\phi}{}^2 + s_{r\phi}{}^2) = 2k^2 ,$$

which with use of equation (A.2.2) may be written as

$$(s_r-s_\theta)^2 + (s_\theta-s_\phi)^2 + (s_\phi-s_r)^2 + 6(s_{r\theta}{}^2+s_{\theta\phi}{}^2+s_{r\phi}{}^2) = 6k^2 ,$$

or, alternatively, as

$$(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_\phi)^2 + (\sigma_\phi-\sigma_r)^2 + 6(\tau_{r\theta}{}^2+\tau_{\theta\phi}{}^2+\tau_{r\phi}{}^2) = 6k^2 .$$

The finite PRANDTL-REUSS constitutive equations (2.4.1a) in terms of physical components of stress deviation and velocity are

(a)

$$\frac{\partial s_r}{\partial t} + v_r \frac{\partial s_r}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_r}{\partial \theta} - 2 s_{r\theta}\right) + \frac{v_\phi}{r\sin\theta} \left(\frac{\partial s_r}{\partial \phi} - 2\sin\theta\, s_{r\theta}\right)$$

$$- \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (r v_\theta)\right] \frac{s_{r\theta}}{r} - \left[\frac{\partial}{\partial r} (r v_\phi) - \frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi}\right] \frac{s_{r\phi}}{r}$$

$$= 2\mu \left(\frac{\partial v_r}{\partial r} - \lambda s_r\right), \qquad \text{(A.2.3a)}$$

(b)

$$\frac{\partial s_\theta}{\partial t} + v_r \frac{\partial s_\theta}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_\theta}{\partial \theta} + 2 s_{r\theta}\right) + \frac{v_\phi}{r\sin\theta} \left(\frac{\partial s_\theta}{\partial \phi} - 2\cos\theta\, s_{\theta\phi}\right)$$

$$- \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta}\right] \frac{s_{r\theta}}{r} - \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta} (\sin\theta v_\phi)\right] \frac{s_{\theta\phi}}{r\sin\theta}$$

$$= 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} - \lambda s_\theta\right), \qquad \text{(A.2.3b)}$$

(c)

$$\frac{\partial s_\phi}{\partial t} + v_r \frac{\partial s_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial s_\phi}{\partial \theta} + \frac{v_\phi}{r} \left(\frac{1}{\sin\theta} \frac{\partial s_\phi}{\partial \phi} + 2 s_{r\phi} + 2\cot\theta\, s_{\theta\phi}\right)$$

$$+ \left[\frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi)\right] \frac{s_{r\phi}}{r} + \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta} (\sin\theta\, v_\phi)\right] \frac{s_{\theta\phi}}{r\sin\theta}$$

$$= 2\mu \left(\frac{1}{r\sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{\cot\theta}{r} v_\theta - \lambda s_\phi\right), \qquad \text{(A.2.3c)}$$

(d)

$$\frac{\partial s_{r\theta}}{\partial t} + v_r \frac{\partial s_{r\theta}}{\partial r} + \frac{v_\theta}{r} \left[\frac{\partial s_{r\theta}}{\partial \theta} + (s_r - s_\theta)\right]$$

$$+ \frac{v_\phi}{r\sin\theta} \left(\frac{\partial s_{r\theta}}{\partial \phi} - \sin\theta\, s_{\theta\phi} - \cos\theta s_{r\phi}\right)$$

$$+ \left(\frac{\partial v_r}{\partial \theta} - r\,\frac{\partial v_\theta}{\partial r} - v_\theta\right) \frac{s_r - s_\theta}{2r} + \left(v_\phi + r\,\frac{\partial v_\phi}{\partial r} - \frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial \phi}\right) \frac{s_{\theta\phi}}{2r}$$

$$- \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}\,(\sin\theta v_\phi)\right] \frac{s_{r\phi}}{2r\sin\theta}$$

$$= 2\mu \left[\frac{1}{2}\left(\frac{1}{r}\,\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) - \lambda s_{r\theta}\right] \qquad \text{(A.2.3d)}$$

(e)

$$\frac{\partial s_{r\phi}}{\partial t} + v_r\,\frac{\partial s_{r\phi}}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_{r\phi}}{\partial \theta} - s_{\theta\phi}\right)$$

$$+ \frac{v_\phi}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial s_{r\phi}}{\partial \phi} + s_r - s_\theta + \cot\theta\, s_{r\theta}\right)$$

$$+ \left[\frac{\partial}{\partial r}\,(r v_\phi) - \frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial \phi}\right] \frac{s_\phi - s_r}{2r} + \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}\,(\sin\theta v_\phi)\right] \frac{s_{r\theta}}{2r}$$

$$- \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}\,(r v_\theta)\right] \frac{s_{\theta\phi}}{2r}$$

$$= 2\mu \left[\frac{1}{2}\left(\frac{1}{r\sin\theta}\,\frac{\partial v_r}{\partial \phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}\right) - \lambda s_{r\phi}\right], \qquad \text{(A.2.3e)}$$

(f)

$$\frac{\partial s_{\theta\phi}}{\partial t} + v_r\,\frac{\partial s_{\theta\phi}}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_{\theta\phi}}{\partial \theta} + s_{r\phi}\right)$$

$$+ \frac{v_\phi}{r\sin\theta}\left[\frac{\partial s_{\theta\phi}}{\partial \phi} + (s_\theta - s_\phi)\cos\theta + s_{r\theta}\sin\theta\right]$$

$$+ \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}\,(r v_\theta)\right] \frac{s_{r\phi}}{2r} - \left[\frac{\partial}{\partial r}\,(r v_\phi) - \frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial \phi}\right] \frac{s_{r\theta}}{2r}$$

$$+ \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}\,(\sin\theta v_\phi)\right] \frac{s_\theta - s_\phi}{2r\sin\theta}$$

$$= 2\mu \left[\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial\theta} - \frac{\cot\theta}{r} v_\phi\right) - \lambda s_{\theta\phi}\right]. \quad \text{(A.2.3f)}$$

In these equations, the joint invariant $\lambda \equiv \frac{1}{2k^2} d^i_j\, s^j_i$, expressed in terms of the physical components of the strain rate and stress deviation tensor, is

$$\lambda = \frac{1}{2k^2}\left[s_r \frac{\partial v_r}{\partial r} + \frac{s_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial\theta}\right) + \frac{s_\phi}{r}\left(v_r + \cot\theta v_\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\right.$$

$$+ \frac{s_{r\theta}}{r}\left(\frac{\partial v_r}{\partial\theta} + r\frac{\partial v_\theta}{\partial r} - v_\theta\right) + \frac{s_{r\phi}}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial v_\phi}{\partial r} - v_\phi\right)$$

$$\left. + \frac{s_{\theta\phi}}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{\partial v_\phi}{\partial\theta} - \cot\theta v_\phi\right)\right].$$

Details of the derivation of the above PRANDTL-REUSS equations are as follows. Rewritten here, equations (2.4.1a), the PRANDTL-REUSS constitutive equations with respect to a general curvilinear coordinate system, are

$$\frac{\mathcal{D}s^{ij}}{\mathcal{D}t} = 2\mu(d^{ij} - \lambda s^{ij})$$

with

$$\frac{\mathcal{D}s^{ij}}{\mathcal{D}t} \equiv \frac{Ds^{ij}}{Dt} - (s^{mj}g^{ki} + s^{im}g^{kj})\omega_{km} ,$$

$$= \frac{\partial s^{ij}}{\partial t} + \left(\frac{\partial s^{ij}}{\partial x^k} + \Gamma^i_{mk}s^{mj} + \Gamma^j_{mk}s^{im}\right)v^k$$

$$- (s^{mj}g^{ki} + s^{im}g^{kj})\omega_{km}.$$

With $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and

(1) $i = 1$, $j = 1$,

$$\frac{Ds^{11}}{Dt} = \frac{\partial s^{11}}{\partial t} + (\frac{\partial s^{11}}{\partial x^k} + \Gamma^1_{mk}s^{m1} + \Gamma^1_{mk}s^{1m})v^k$$

$$= \frac{\partial s^{11}}{\partial t} + (\frac{\partial s^{11}}{\partial x^1} + 2\Gamma^1_{m1}s^{m1})v^1 + (\frac{\partial s^{11}}{\partial x^2} + 2\Gamma^1_{m2}s^{m1})v^2$$

$$+ (\frac{\partial s^{11}}{\partial x^3} + 2\Gamma^1_{m3}s^{m1})v^3$$

$$= \frac{\partial s^{11}}{\partial t} + \frac{\partial s^{11}}{\partial x^1} v^1 + v^2 (\frac{\partial s^{11}}{\partial x^2} - 2x^1 s^{12}) + v^3 (\frac{\partial s^{11}}{\partial x^3} - 2x^1 \sin^2 x^2 s^{13})$$

$$= \frac{\partial s_r}{\partial t} + v_r \frac{\partial s_r}{\partial r} + \frac{v_\theta}{r} (\frac{\partial s_r}{\partial \theta} - 2s_{r\theta}) + \frac{v_\phi}{r\sin\theta} (\frac{\partial s_r}{\partial \phi} - 2\sin\theta\ s_{r\phi}),$$

$$- (s^{m1}g^{k1} + s^{1m}g^{k1})\omega_{km} = - 2s^{21}g^{11}\omega_{12} - 2s^{13}g^{11}\omega_{13}$$

$$= - \frac{s_{r\theta}}{r} [\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (rv_\theta)] - \frac{s_{r\phi}}{r} [\frac{\partial}{\partial r} (rv_\phi) - \frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi}],$$

$$2\mu(d^{11} - \lambda s^{11}) = 2\mu(\frac{\partial v_r}{\partial r} - \lambda s_r),$$

combine to yield equation (A.2.3a);

(2) $i = 2$, $j = 2$,

$$\frac{Ds^{22}}{Dt} = \frac{\partial}{\partial t} s^{22} + (\frac{\partial s^{22}}{\partial x^1} + 2\Gamma^2_{m1}s^{2m})v^1 + (\frac{\partial s^{22}}{\partial x^2} + 2\Gamma^2_{m2}s^{2m})v^2$$

$$+ \left(\frac{\partial s^{22}}{\partial x^3} + 2\Gamma^2_{m3} s^{2m}\right) v^3$$

$$= \frac{\partial}{\partial t} s^{22} + \left(\frac{\partial s^{22}}{\partial x^1} + \frac{2}{x^1} s^{22}\right) v^1 + \left(\frac{\partial s^{22}}{\partial x^2} + \frac{2}{x^1} s^{21}\right) v^2$$

$$+ \left(\frac{\partial s^{22}}{\partial x^3} - 2 \sin x^2 \cos x^2 s^{23}\right) v^3$$

$$= \frac{1}{r^2} \frac{\partial s_\theta}{\partial t} + \left[\frac{\partial}{\partial r} \left(\frac{s_\theta}{r^2}\right) + \frac{2s_\theta}{r^3}\right] v_r + \left[\frac{\partial}{\partial \theta} \left(\frac{s_\theta}{r^2}\right) + \frac{2s_{r\theta}}{r^2}\right] \frac{v_\theta}{r}$$

$$+ \left[\frac{\partial}{\partial \phi} \left(\frac{s_\theta}{r^2}\right) - \frac{2\cos\theta}{r^2} s_{\theta\phi}\right] \frac{v_\phi}{r\sin\theta} ,$$

$$- (s^{m2} g^{k2} + s^{2m} g^{k2}) \omega_{km} = - 2s^{m2} g^{22} \omega_{2m}$$

$$= - \frac{2}{(x^1)^2} (s^{12}\omega_{21} + s^{32}\omega_{23})$$

$$= - \frac{1}{r^3} \left\{s_{r\theta}\left[\frac{\partial}{\partial r} (rv_\theta) - \frac{\partial v_r}{\partial \theta}\right] + \frac{s_{\theta\phi}}{\sin\theta} \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta} (\sin\theta\, v_\phi)\right]\right\},$$

$$2\mu \, (d^{22} - \lambda s^{22}) = \frac{2\mu}{r^2} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} - \lambda s_\theta\right)$$

combine to form equation (A.2.3b);

(3) $i = 3, j = 3,$

$$\frac{Ds^{33}}{Dt} = \frac{\partial s^{33}}{\partial t} + \left(\frac{\partial s^{33}}{\partial x^1} + 2\Gamma^3_{m1} s^{m3}\right) v^1 + \left(\frac{\partial s^{33}}{\partial x^2} + 2\Gamma^3_{m2} s^{m3}\right) v^2$$

$$+ \left(\frac{\partial s^{33}}{\partial x^3} + 2\Gamma^3_{m3} s^{m3}\right) v^3$$

$$= \frac{\partial s^{33}}{\partial t} + \left(\frac{\partial s^{33}}{\partial x^1} + \frac{2}{x^1} s^{33}\right) v^1 + \left(\frac{\partial s^{33}}{\partial x^2} + 2 \cot(x^2) s^{33}\right) v^2$$

$$+ \left(\frac{\partial s^{33}}{\partial x^3} + \frac{2}{x^1} s^{13} + 2 \cot(x^2) s^{23}\right) v^3$$

$$= \frac{1}{r^2 \sin^2\theta} \left[\frac{\partial s_\phi}{\partial t} + v_r \frac{\partial s_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial s_\phi}{\partial \theta}\right.$$

$$\left. + \frac{v_\phi}{r} \left(\frac{1}{\sin\theta} \frac{\partial s_\phi}{\partial \phi} + 2 s_{r\phi} + 2 \cot\theta\, s_{\theta\phi}\right)\right],$$

$$- (s^{m3} g^{k3} + s^{3m} g^{k3}) \omega_{km} = - 2 s^{m3} g^{33} \omega_{3m}$$

$$= - 2 s^{13} g^{33} \omega_{31} - 2 s^{23} g^{33} \omega_{32}$$

$$= \frac{2}{r^2 \sin^2\theta} \left\{\frac{s_{r\phi}}{2} \left[\frac{1}{r\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r v_\phi)\right]\right.$$

$$\left. + \frac{s_{\theta\phi}}{2r\sin\theta} \left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta} (\sin\theta\, v_\phi)\right]\right\}$$

$$2\mu \left(d^{33} - \lambda s^{33}\right) = 2\mu \left(\frac{d_\phi}{r^2 \sin^2\theta} - \frac{s_\phi}{r^2 \sin^2\theta}\right)$$

$$= \frac{2\mu}{r^2 \sin^2\theta} \left(\frac{1}{r\sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{\cot\theta}{r} v_\theta - \lambda s_\phi\right)$$

combine to form equation (A.2.3c);

(4) i = 1, j = 2,

$$\frac{Ds^{12}}{Dt} = \frac{\partial s^{12}}{\partial t} + (\frac{\partial s^{12}}{\partial x^1} + \Gamma^2_{m1} s^{1m}) v^1 + (\frac{\partial s^{12}}{\partial x^2} + \Gamma^1_{m2} s^{m2} + \Gamma^2_{m2} s^{1m}) v^2$$

$$+ (\frac{\partial s^{12}}{\partial x^3} + \Gamma^1_{m3} s^{m2} + \Gamma^2_{m3} s^{1m}) v^3$$

$$= \frac{\partial s^{12}}{\partial t} + (\frac{\partial s^{12}}{\partial x^1} + \frac{1}{x^1} s^{12}) v^1 + (\frac{\partial s^{12}}{\partial x^2} + (-x^1)\ s^{22} + \frac{1}{x^1} s^{11}) v^2$$

$$+ (\frac{\partial s^{12}}{\partial x^3} - x^1 \sin^2 x^2) s^{32} + (-\sin x^2 \cos x^2 s^{13}) v^3$$

$$= \frac{\partial}{\partial t} (\frac{s_{r\theta}}{r}) + [\frac{\partial}{\partial r} (\frac{s_{r\theta}}{r}) + \frac{s_{r\theta}}{r^2}]\ v_r + [\frac{\partial}{\partial \theta} (\frac{s_{r\theta}}{r}) - \frac{s_\theta}{r} + \frac{s_r}{r}]\ \frac{v_\theta}{r}$$

$$+ [\frac{\partial}{\partial \phi} (\frac{s_{r\theta}}{r}) - \frac{\sin\theta}{r}\ s_{\theta\phi} - \cos\theta\ \frac{s_{r\theta}}{r}]\ \frac{v_\phi}{r\sin\theta}\ ,$$

$$- (s^{m2} g^{k1} + s^{1m} g^{k2}) \omega_{km}$$

$$= - (s^{12} g^{11} + s^{11} g^{k2}) \omega_{k1} - (s^{22} g^{k1} + s^{12} g^{k2}) \omega_{k2}$$

$$- (s^{32} g^{k1} + s^{13} g^{k2}) \omega_{k3}$$

$$= - s^{11} g^{22} \omega_{21} - s^{22} g^{11} \omega_{12} - s^{32} g^{11} \omega_{13} - s^{13} g^{22} \omega_{23}$$

$$= \frac{1}{2r^2}\ \{(s_r - s_\theta)\ [\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (r v_\theta)]$$

$$+ s_{\theta\phi} [\frac{\partial}{\partial r} (rv_{\phi}) - \frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi}]$$

$$- s_{r\phi} \sin\theta [\frac{\partial v_{\theta}}{\partial \phi} - \frac{\partial}{\partial \theta} (\sin\theta v_{\phi})],$$

$$2\mu (d^{12} - \lambda s^{12}) = \frac{2\mu}{r} [\frac{1}{2} (\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_{\theta}}{\partial r} - \frac{v_{\theta}}{r}) - \lambda s_{r\theta}]$$

combine to form equation (A.2.3d);

(5) $i = 1$, $j = 3$,

$$\frac{Ds^{13}}{Dt} = \frac{\partial s^{13}}{\partial t} + (\frac{\partial s^{13}}{\partial x^1} + \Gamma^3_{m1} s^{1m}) v^1 + (\frac{\partial s^{13}}{\partial x^2} + \Gamma^1_{m2} s^{m3} + \Gamma^3_{m2} s^{1m}) v^2$$

$$+ (\frac{\partial s^{13}}{\partial x^3} + \Gamma^1_{m3} s^{m3} + \Gamma^3_{m3} s^{1m}) v^3$$

$$= \frac{\partial s^{13}}{\partial t} + (\frac{\partial s^{13}}{\partial x^1} + \frac{1}{x^1} s^{13}) v^1 + (\frac{\partial s^{13}}{\partial x^2} - x^1 s^{23} + \cot x^2 s^{13}] v^2$$

$$+ (\frac{\partial s^{13}}{\partial x^3} - x^1 \sin^2 x^2 s^{33} + \frac{1}{x^1} s^{11} + \cot x^2 s^{12}) v^3$$

$$= \frac{1}{r\sin\theta} [\frac{\partial s_{r\theta}}{\partial t} + v_r \frac{\partial s_{r\phi}}{\partial r} + \frac{v_{\theta}}{r} (\frac{\partial s_{r\phi}}{\partial \theta} - s_{\theta\phi})$$

$$+ \frac{v_{\phi}}{r} (\frac{1}{\sin\theta} \frac{\partial s_{r\phi}}{\partial \phi} + s_r - s_{\theta} + \cot\theta \, s_{r\theta})],$$

$$- (s^{m3} g^{k1} + s^{1m} g^{k3}) \omega_{km} = - s^{11} g^{33} \omega_{31} - s^{23} g^{11} \omega_{12} - s^{12} g^{33} \omega_{32} - s^{33} g^{11} \omega_{13}$$

$$= \frac{1}{2r\sin\theta}\{\frac{s_\phi - s_r}{r}[\frac{\partial}{\partial r}(rv_\phi) - \frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}]$$

$$+ \frac{s_{r\theta}}{r}[\frac{\partial v_\theta}{\partial\phi} - \frac{\partial}{\partial\theta}(\sin\theta v_\phi)]$$

$$- \frac{s_{\theta\phi}}{r}[\frac{\partial v_r}{\partial\theta} - \frac{\partial}{\partial r}(rv_\theta)],$$

$$2\mu\,(d^{13} - \lambda s^{13}) = \frac{2\mu}{r\sin\theta}[\frac{1}{2}(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}) - \lambda s_{r\phi}]$$

combine to form equation (A.2.3e); and for

(6) $i = 2$, $j = 3$,

$$\frac{Ds^{23}}{Dt} = \frac{\partial s^{23}}{\partial t} + (\frac{\partial s^{23}}{\partial x^1} + \Gamma^2_{m1}s^{m3} + \Gamma^3_{m1}s^{2m})v^1$$

$$+ (\frac{\partial s^{23}}{\partial x^2} + \Gamma^2_{m2}s^{m3} + \Gamma^3_{m2}s^{2m})v^2$$

$$+ (\frac{\partial s^{23}}{\partial x^3} + \Gamma^2_{m3}s^{m3} + \Gamma^2_{m3}s^{2m})v^3$$

$$= \frac{\partial s^{23}}{\partial^t} + (\frac{\partial s^{23}}{\partial x^1} + \frac{1}{x^1}s^{23} + \frac{1}{x^1}s^{23})v^1$$

$$+ (\frac{\partial s^{23}}{\partial x^2} + \frac{1}{x^1}s^{13} + \cot x^2 s^{23})v^2$$

$$+ (\frac{\partial s^{23}}{\partial x^3} - \sin x^2\cos x^2 s^{33} + \frac{1}{x^1}s^{21} + \cot x^2 s^{22})v^3$$

$$= \frac{1}{r^2\sin\theta}\,[\frac{\partial s_{\theta\phi}}{\partial t} + v_r\,\frac{\partial s_{\theta\phi}}{\partial r} + (\frac{\partial s_{\theta\phi}}{\partial\theta} + s_{r\phi})\,\frac{v_\theta}{r}$$

$$+ (\frac{\partial s_{\theta\phi}}{\partial\phi} + (s_\theta - s_\phi)\cos\theta + s_{r\theta}\sin\theta)\,\frac{v_\phi}{r\sin\theta}] \,,$$

$$- (s^{m2}g^{k3} + s^{3m}g^{k2})\omega_{km} = - s^{31}g^{22}\omega_{21} - s^{12}g^{33}\omega_{31}$$

$$- s^{22}g^{33}\omega_{32} - s^{33}g^{22}\omega_{23}$$

$$= \frac{1}{2r^3\sin\theta}\,\{s_{r\phi}\,[\frac{\partial v_r}{\partial\theta} - \frac{\partial}{\partial r}\,(rv_\theta)] - s_{r\theta}[\frac{\partial}{\partial r}\,(rv_\phi) - \frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial\phi}]$$

$$+ \frac{s_\theta - s_\phi}{\sin\theta}\,[\frac{\partial v_\theta}{\partial\phi} - \frac{\partial}{\partial\theta}\,(\sin\theta v_\phi)],$$

and

$$2\mu\,(d^{23} - \lambda s^{23}) = \frac{2\mu}{r^2\sin\theta}\,[\frac{1}{2}\,(\frac{1}{r\sin\theta}\,\frac{\partial v_\theta}{\partial\phi} + \frac{1}{r}\,\frac{\partial v_\phi}{\partial\theta} - \frac{\cot\theta}{r}\,v_\phi)$$

$$- \lambda s_{\theta\phi}]$$

combine to form equation (A.2.3f).

A.3 CHRISTOFFEL SYMBOLS AND PHYSICAL COMPONENTS FOR A CYLINDRICAL POLAR COORDINATE SYSTEM

For a cylindrical polar coordinate system r, θ,z as illustrated in Figure 2, the line element is given by

$$ds^2 = dr^2 + r^2d\theta^2 + dz^2.$$

The covariant and contravariant components of the metric tensor are

$$g_{11} = 1, \; g_{22} = r^2, \; g_{33} = 1, \; g_{ij} = 0 \; (i \neq j),$$

and

$$g^{11} = 1, \; g^{22} = \frac{1}{r^2}, \; g^{33} = 1, \; g^{ij} = 0 \; (i \neq j);$$

and the non-zero CHRISTOFFEL symbols of the second kind are

$$\Gamma^1_{22} = -r, \; \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} .$$

In terms of the corresponding covariant and contravariant components, the physical components of velocity are

$$v_r = v_1 = v^1, \; v_\theta = \frac{v_2}{r} = rv^2, \; v_z = v_3 = v^3,$$

and the physical components of the symmetric stress tensor are

$$\sigma_r = \sigma_{11} = \sigma^{11}, \; \sigma_\theta = \frac{\sigma_{22}}{r^2} = r^2\sigma^{22}, \; \sigma_3 = \sigma_{33} = \sigma^{33},$$

$$\tau_{r\theta} = \frac{\sigma_{12}}{r} = r\sigma^{12}, \; \tau_{rz} = \sigma_{13} = \sigma^{13}, \; \tau_{\theta z} = \frac{\sigma_{23}}{r} = r\sigma^{23},$$

with similar relationships holding between the covariant, contravariant and physical components s_r, s_θ, s_z, $s_{r\theta}$, s_{rz}, $s_{\theta z}$ of the stress deviation tensor. Also

$$s^{ij} = \sigma^{ij} + pg^{ij}, \; p = -\frac{1}{3} g_{mn}\sigma^{mn}$$

gives

$$s_r = \sigma_r + p,\ s_\theta = \sigma_\theta + p,\ s_z = \sigma_z + p,$$

$$s_{r\theta} = \tau_{r\theta},\ s_{rz} = \tau_{rz},\ s_{\theta z} = \tau\theta_z,$$

with $p = -\frac{1}{3}(\sigma_r + \sigma_\theta + \sigma_z)$.

The physical components of the symmetric strain rate tensor are

$$d_r = \frac{\partial v_r}{\partial r},\ d_\theta = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r},\ d_z = \frac{\partial v_z}{\partial z},$$

$$d_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),\ d_{rz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)$$

$$d_{\theta z} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right),$$

and the physical components of the skew-symmetric vorticity tensor are

$$\omega_{r\theta} = \frac{1}{2r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right],\ \omega_{rz} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right),$$

$$\omega_{\theta z} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right).$$

A.4 GOVERNING EQUATIONS IN CYLINDRICAL POLAR COORDINATES

The equations of motion (2.4.2) in terms of the physical components of stress and velocity are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho\left(\frac{Dv_r}{Dt} - F_r\right),$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} = \rho\left(\frac{Dv_\theta}{Dt} - F_\theta\right),$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho\left(\frac{Dv_z}{Dt} - F_z\right),$$

where

$$\frac{Dv_r}{Dt} = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^{\,2}}{r},$$

$$\frac{Dv_\theta}{Dt} = \frac{\partial v_\theta}{\partial t} + \frac{v_r}{r}\frac{\partial}{\partial r}(rv_\theta) + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z},$$

$$\frac{Dv_z}{Dt} = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z},$$

ρ is the density and F_r, F_θ, F_z are the physical components of the body force per unit mass. For quasi-static problems in the absence of body forces, the right hand sides of equations (A.4.1a,b,c) are zero and the equations are called the equations of equilibrium.

The equation of continuity (2.4.3) in terms of the physical components of velocity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0.$$

Also if the elastic-perfectly plastic solid is incompressible, there is the equation of incompressibility

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0.$$

The VON MISES yield condition (2.4.4) is

$$s_r^2 + s_\theta^2 + s_z^2 + 2(s_{r\theta}^2 + s_{rz}^2 + s_{\theta z}^2) = 2k^2 \tag{A.4.2}$$

and the identity (2.4.5) is

$$s_r + s_\theta + s_z \equiv 0. \tag{A.4.3}$$

The PRANDTL-REUSS constitutive equations (2.4.1a) in terms of the components of stress deviation and velocity are

(a)

$$\frac{\partial s_r}{\partial t} + v_r \frac{\partial s_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_r}{\partial \theta} - 2s_{r\theta}\right) + v_z \frac{\partial s_r}{\partial z} - \frac{s_{r\theta}}{r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right]$$

$$- \frac{s_{rz}}{r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right] = 2\mu\left(\frac{\partial v_r}{\partial r} - \lambda s_r\right), \tag{A.4.4a}$$

(b)

$$\frac{\partial s_\theta}{\partial t} + v_r \frac{\partial s_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_\theta}{\partial \theta} + 2s_{r\theta}\right) + v_z \frac{\partial s_\theta}{\partial z}$$

$$+ \frac{s_{r\theta}}{r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right] - s_{\theta z}\left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$= 2\mu\left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) - \lambda s_\theta\right], \tag{A.4.4b}$$

(c)

$$\frac{\partial s_z}{\partial t} + v_r \frac{\partial s_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial s_z}{\partial \theta} + v_z \frac{\partial s_z}{\partial z} + s_{rz}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + s_{\theta z}\left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$= 2\mu\left[\frac{\partial v_z}{\partial z} - \lambda s_z\right], \tag{A.4.4c}$$

(d)

$$\frac{\partial s_{r\theta}}{\partial t} + v_r \frac{\partial s_{r\theta}}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_{r\theta}}{\partial \theta} + s_r - s_\theta\right)$$

$$+ \frac{s_r - s_\theta}{2r} \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (r v_\theta)\right] - \frac{s_{\theta z}}{2} \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)$$

$$- \frac{s r_z}{2} \left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right) = 2\mu \left[\frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) - \lambda s_{r\theta}\right] \quad \text{(A.4.4d)}$$

(e)

$$\frac{\partial s_{rz}}{\partial t} + v_r \frac{\partial s_{rz}}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_{rz}}{\partial \theta} - s_{\theta z}\right) + v_z \frac{\partial s_{rz}}{\partial z}$$

$$- \frac{s_{\theta z}}{2r} \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (r v_\theta)\right] + \frac{s_{r\theta}}{2} \left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right)$$

$$+ \frac{s_r - s_z}{2} \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = 2\mu\left[\frac{1}{2} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right) - \lambda s_{rz}\right] , \qquad \text{(A.4.4e)}$$

(f)

$$\frac{\partial s_{\theta z}}{\partial t} + v_r \frac{\partial s_{\theta z}}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_{\theta z}}{\partial \theta} + s_{rz}\right) + v_z \frac{\partial s_{\theta z}}{\partial z}$$

$$+ \frac{s_{rz}}{2r} \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (r v_\theta)\right] + \frac{s_{r\theta}}{2} \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)$$

$$+ \frac{s_\theta - s_z}{2} \left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right) = 2\mu\left[\frac{1}{2} \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right) - \lambda s_{\theta z}\right]. \quad \text{(A.4.4f)}$$

where the joint invariant $\lambda \equiv \frac{1}{2k^2} d^i_j s^j_i$ is

$$\lambda = \frac{1}{2k^2} \left[s_r \frac{\partial v_r}{\partial r} + \frac{s_\theta}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + s_z \frac{\partial v_z}{\partial z}\right.$$

$$+ \frac{s_{r\theta}}{r} \left(\frac{\partial v_r}{\partial \theta} + r \frac{\partial v_\theta}{\partial r} v_\theta\right) + s_{rz} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)$$

$$+ s_{\theta z} \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)].$$

Details of the derivation of the above PRANDTL-REUSS equations are as follows. Proceeding as in SECTION A.2 above with $x^1 = r$, $x^2 = \theta$ and $x^3 = z$ and

(1) $i = 1$, $j = 1$,

$$\frac{Ds^{11}}{Dt} = \frac{\partial s^{11}}{\partial t} + \frac{\partial s^{11}}{\partial x^1} v^1 + \left(\frac{\partial s^{11}}{\partial x^2} + \Gamma^1_{22} s^{21} + \Gamma^1_{22} s^{12}\right) v^2 + \frac{\partial s^{11}}{\partial x^3} v^3$$

$$= \frac{\partial s_r}{\partial t} + v_r \frac{\partial s_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_r}{\partial \theta} - 2s_{r\theta}\right) + v_z \frac{\partial s_r}{\partial z},$$

$$-(s^{m1}g^{k1} + s^{1m}g^{k1})\omega_{km} = -2s^{1m}g^{11}\omega_{1m}$$

$$= -2s_{r\theta}\left[\frac{1}{2r}\left\{\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right\}\right] - \frac{s_{rz}}{r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(rv_\theta)\right],$$

$$2\mu(d^{11} - \lambda s^{11}) = 2\mu\left(\frac{\partial v_r}{\partial r} - \lambda s_r\right)$$

combine to form equation (A.4.4a);

(2) $i = 2$, $j = 2$,

$$\frac{Ds^{22}}{Dt} = \frac{\partial s^{22}}{\partial t} + \left(\frac{\partial s^{22}}{\partial x^1} + 2\Gamma^2_{21}\right) v^1 + \left(\frac{\partial s^{22}}{\partial x^2} + 2\Gamma^2_{12} s^{12}\right) v^2 + \frac{\partial s^{22}}{\partial x^3} v^3$$

$$= \frac{1}{r^2}\left[\frac{\partial s_\theta}{\partial t} + v_r \frac{\partial s_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial s_\theta}{\partial \theta} + 2s_{r\theta}\right) + v_z \frac{\partial s_\theta}{\partial z}\right],$$

$$- (s^{m2}g^{k2} + s^{2m}g^{k2})\omega_{km} = - 2s^{2m}g^{22}\omega_{2m}$$

$$= - \frac{2}{r^2} [- \frac{s_{r\theta}}{2r} \{\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (rv_\theta)\} + \frac{s_{\theta z}}{2r} (\frac{\partial v_\theta}{\partial \theta} - \frac{1}{r} \frac{\partial v_z}{\partial \theta})],$$

$$2\mu(d^{22} - \lambda s^{22}) = \frac{2\mu}{r^2} (\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} - \lambda s_\theta)$$

combine to form equation (A.4.4b);

(3) i = 3, j = 3,

$$\frac{Ds^{33}}{Dt} = \frac{\partial s^{33}}{\partial t} + \frac{\partial s^{33}}{\partial x^k} v^k$$

$$= \frac{\partial s_z}{\partial t} + v_r \frac{\partial s_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial s_z}{\partial \theta} + v_z \frac{\partial s_z}{\partial z},$$

$$- (s^{m3}g^{k3} + s^{3m}g^{k3})\omega_{km} = - 2s^{m3}g^{33}\omega_{3m}$$

$$= s_{rz} (\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}) + s_{\theta z} (\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}),$$

$$2\mu(d^{33} - \lambda s^{33}) = 2\mu(\frac{\partial v_z}{\partial z} - \lambda s_z)$$

combine to form equation (A.4.4c);

(4) i = 1, j = 2,

$$\frac{Ds^{12}}{Dt} = \frac{\partial s^{12}}{\partial t} + (\frac{\partial s^{12}}{\partial x^1} + \Gamma^1_{21} s^{12})v^1$$

$$+ \left(\frac{\partial s^{12}}{\partial x^2} + \Gamma^1_{22}s^{22} + \Gamma^2_{12}s^{11}\right)v^2 + \frac{\partial s^{12}}{\partial x^3} v^3$$

$$= \frac{1}{r} \left(\frac{\partial s_{r\theta}}{\partial t} + v_r \frac{\partial s_{r\theta}}{\partial r} + \frac{v_\theta}{r} \frac{\partial s_{r\theta}}{\partial \theta} + v_z \frac{\partial s_{r\theta}}{\partial z} + \frac{s_r - s_\theta}{r} v_\theta\right),$$

$$- (s^{m2}g^{k1} + s^{1m}g^{k2})\omega_{km} = - (s^{m2}g^{11}\omega_{1m} + s^{1m}g^{22}\omega_{2m}$$

$$= \frac{1}{r} \left[\frac{s_r - s_\theta}{2r} \left\{\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (rv_\theta)\right\} + \frac{s_{\theta z}}{2} \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\right.$$

$$\left. + \frac{s_{rz}}{2} \left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right)\right],$$

$$2\mu(d^{12} - \lambda s^{12}) = \frac{2\mu}{r} \left[\frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) - \lambda s_{r\theta}\right]$$

combine to form equation (A.4.4d);

(5) $i = 1$, $j = 3$,

$$\frac{Ds^{13}}{Dt} = \frac{\partial s^{13}}{\partial t} + v^1 \frac{\partial s^{13}}{\partial x^1} + \left(\frac{\partial s^{13}}{\partial x^2} + \Gamma^1_{22}s^{23}\right)v^2 + \frac{\partial s^{13}}{\partial x^3} v^3$$

$$= \frac{\partial s_{rz}}{\partial t} + v_r \frac{\partial s_{rz}}{\partial r} + \frac{v_\theta}{r} \left(\frac{\partial s_{rz}}{\partial \theta} - s_{\theta z}\right) + \frac{\partial s_{rz}}{\partial z} v_z ,$$

$$- (s^{m3}g^{k1} + s^{1m}g^{k3})\omega_{km} = - s^{23}g^{11}\omega_{12} - s^{33}g^{11}\omega_{13} - s^{12}g^{33}\omega_{32} - s^{11}g^{22}\omega_{31}$$

$$= - \frac{s_{\theta z}}{2r} \left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r} (rv_\theta)\right] + \frac{s_{r\theta}}{2} \left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r} \frac{\partial v_z}{\partial \theta}\right)$$

$$+ \frac{s_r - s_z}{2} \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right),$$

$$2\mu(d^{13} - \lambda s^{13}) = 2\mu[\frac{1}{2}(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}) - \lambda s_{rz}]$$

combine to form (A.4.4e); and for

(6) i = 2, j = 3,

$$\frac{Ds^{23}}{Dt} = \frac{\partial s^{23}}{\partial t} + (\frac{\partial s^{23}}{\partial x^1} + \Gamma^2_{m1} s^{m3})v^1 + (\frac{\partial s^{23}}{\partial x^2} + \Gamma^2_{m2} s^{m3})\ v^2$$

$$+ \frac{\partial s^{23}}{\partial x^3}\ v^3$$

$$= \frac{\partial s^{23}}{\partial t} + (\frac{\partial s^{23}}{\partial x^1} + \Gamma^2_{21} s^{23})v^1 + (\frac{\partial s^{23}}{\partial x^2} + \Gamma^2_{12} s^{13})v^2 + \frac{\partial s^{23}}{\partial x^3}\ v^3$$

$$= \frac{1}{r}\ [\frac{\partial s_{\theta z}}{\partial t} + v_r\ \frac{\partial s_{\theta z}}{\partial r} + \frac{v_\theta}{r}\ (\frac{\partial s_{\theta z}}{\partial \theta} + \frac{s_{rz}-s_{\theta z}}{r}) + v_z\ \frac{\partial s_{\theta z}}{\partial z}]\ ,$$

$$-\ (s^{m3}g^{k2} + s^{2m}g^{k3})\omega_{km} = -\ s^{m3}g^{22}\omega_{2m} - s^{2m}g^{33}\omega_{3m}$$

$$= -\ s^{13}g^{22}\omega_{21} - s^{33}g^{22}\omega_{23} - s^{21}g^{33}\omega_{31} - s^{22}g^{33}\omega_{32}$$

$$= \frac{1}{r}\ [\frac{s_{rz}}{2}\ \{\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}\ (rv_\theta)\} + \frac{s_{r\theta}}{2}\ (\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r})$$

$$+ \frac{s_\theta - s_z}{2}\ (\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\ \frac{\partial v_z}{\partial \theta})],$$

$$2\mu(d^{23} - \lambda s^{23}) = \frac{2\mu}{r}[\frac{1}{2}\ (\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\ \frac{\partial v_z}{\partial \theta}) - \lambda s_{\theta z})$$

combine to form equation (A.4.4f).

APPENDIX B

CHARACTERISTIC STUDY FOR PLANE STRAIN FLOW OF AN INCOMPRESSIBLE ELASTIC-PERFECTLY PLASTIC MISES SOLID

In this appendix, a supplementary study of the characteristics of the governing equations for steady state, quasi-static plane plastic flow of an incompressible elastic-perfectly plastic MISES solid is given using rectangular CARTESIAN coordinates (x,y,z). The procedure is different from that used in CHAPTER VI, SECTION 6.1 and alternate forms for the compatibility relationships are derived.

B.1 GOVERNING EQUATIONS IN RECTANGULAR CARTESIAN COORDINATES

For plane strain flow independent of z and parallel to the (x,y)-plane

$$\tau_{xz} = \tau_{yz} = v_z = d_{xz} = d_{yz} = d_z = 0 .$$

Since $d_z = 0$, $s_z = 0$ and σ_z is equal to the hydrostatic pressure -p. Consequently,

$$s_x = - s_y \qquad \text{(B.1.1)}$$

and the VON MISES yield criterion is

$$s_x^2 + s_{xy}^2 = k^2$$

where k is the yield stress in pure shear. s_x, s_y and s_{xy} are stress

deviation components.

The governing equations used with appropriate boundary conditions to determine s_x, s_{xy}, $-p$ and the velocity components v_x and v_y are the PRANDTL-REUSS equation

$$\frac{k^2}{\mu}\left(v_x \frac{\partial s_x}{\partial x} + v_y \frac{\partial s_x}{\partial y}\right) - (2k^2 - s_x{}^2)\frac{\partial v_x}{\partial x} - \left(\frac{k^2}{\mu} - s_x\right) s_{xy} \frac{\partial v_x}{\partial y}$$

$$+ \left(\frac{k^2}{\mu} + s_x\right) s_{xy} \frac{\partial v_y}{\partial x} - s_x{}^2 \frac{\partial v_y}{\partial y} = 0 \; ; \qquad \text{(B.1.3a)}$$

the equations of equilibrium

$$\frac{\partial s_x}{\partial x} + \frac{\partial s_{xy}}{\partial y} - \frac{\partial p}{\partial y} = 0 \, , \qquad \text{(B.1.3b)}$$

$$\frac{\partial s_x}{\partial y} - \frac{\partial s_{xy}}{\partial x} + \frac{\partial p}{\partial y} = 0; \qquad \text{(B.1.3c)}$$

the equation of incompressibility

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0; \qquad \text{(B.1.3d)}$$

and the VON MISES yield criterion (B.1.2). Equation (B.1.3a) is obtained from equation (2.4.1a) by putting $i = j = 1$ and $x_1 = x$, $x_2 = y$, $x_3 = z$ and noting that rectangular CARTESIAN tensor components are physical components.

The yield criterion (B.1.2) is satisfied identically on defining

$$s_x \equiv - k \sin 2\phi \qquad \text{(B.1.4a)}$$

and

$$s_y \equiv k \cos 2\phi \tag{B.1.4b}$$

where $\phi + \frac{\pi}{4}$ is the angle between the direction of the algebraically greater principal stress in the (x,y) - plane and the positive x-axis. Substitution of equations (B.1.4a,b) into equations (B.1.3a,b,c) together with use of equation (B.1.3d) yields respectively

$$2 \frac{k}{\mu} (v_x \frac{\partial \phi}{\partial x} + v_y \frac{\partial \phi}{\partial y}) + 2 \cos 2\phi \frac{\partial v_x}{\partial x}$$

$$+ (\frac{k}{\mu} + \sin 2\phi) \frac{\partial v_x}{\partial y} - (\frac{k}{\mu} - \sin 2\phi) \frac{\partial v_y}{\partial x} = 0, \qquad \text{(B.1.5a,b,c)}$$

$$\cos 2\phi \frac{\partial \phi}{\partial x} + \sin 2\phi \frac{\partial \phi}{\partial y} + \frac{\partial P}{\partial x} = 0,$$

and

$$\sin 2\phi \frac{\partial \phi}{\partial x} - \cos 2\phi \frac{\partial \phi}{\partial y} + \frac{\partial P}{\partial y} = 0,$$

where $P \equiv \frac{p}{2k}$. Considering only the case where $0 < \frac{k}{\mu} < 1$ and defining $\frac{k}{\mu} \equiv \sin 2\gamma$, equation (B.1.5a) can be written as

$$2 \sin 2\gamma \, (v_x \frac{\partial \phi}{\partial x} + v_y \frac{\partial \phi}{\partial y}) + 2 \cos 2\phi \frac{\partial v_x}{\partial x}$$

$$+ (\sin 2\gamma + \sin 2\phi) \frac{\partial v_x}{\partial y} - (\sin 2\gamma - \sin 2\phi) \frac{\partial v_y}{\partial x} = 0. \tag{B.1.6}$$

The four governing equations used to determine the unknowns ϕ, P, v_x and

v_y are equations (B.1.6), (B.1.5b,c) and (B.1.3d). For the determination of any existing characteristics these governing equations are supplemented with the relations

$$\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy = dP,$$

$$\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = d\phi, \qquad \text{(B.1.7a,b,c,d)}$$

$$\frac{\partial v_x}{\partial x}\,dx + \frac{\partial v_x}{\partial y}\,dy = dv_x,$$

$$\frac{\partial v_y}{\partial x}\,dx + \frac{\partial v_y}{\partial y}\,dy = dv_y.$$

B.2 DETERMINATION OF CHARACTERISTICS AND COMPATIBILITY RELATIONSHIPS

Let Γ be any member of the family of curves in the (x,y) - plane governed by the differential equation

$$\frac{dy}{dx} = \tan\zeta,\ \zeta \equiv \zeta(x,y),$$

where $dx = \cos\zeta\,ds$, $dy = \sin\zeta\,ds$ and ds is the elemental arclength along Γ. Along Γ it follows that

$$\frac{dP}{ds} = -\cos(2\phi - \zeta)\,\frac{\partial \phi}{\partial x} - \sin(2\phi - \zeta)\,\frac{\partial \phi}{\partial y} \qquad \text{(B.2.1)}$$

with use of equations (B.1.5b,c) and (B.1.7a) and that

$$\frac{d\phi}{ds} = \cos\zeta\,\frac{\partial \phi}{\partial x} + \sin\zeta\,\frac{\partial \phi}{\partial y} \qquad \text{(B.2.2)}$$

with the use of equation (B.1.7b). From equations (B.2.1) and (B.2.2),

$$\frac{\partial \phi}{\partial x} = \frac{1}{\sin 2(\phi-\zeta)} [\sin(2\phi-\zeta) \frac{d\phi}{ds} + \sin\zeta \frac{dP}{ds}] ,$$

(B.2.3a,b)

$$\frac{\partial \phi}{\partial y} = \frac{1}{\sin 2(\phi- \zeta)} [- \cos(2\phi-\zeta) \frac{d\phi}{ds} - \cos \zeta \frac{dP}{ds}] ,$$

and hence from equations (B.1.5b,c),

$$\frac{\partial P}{\partial x} = \frac{1}{\sin 2(\phi-\zeta)} [\sin \zeta \frac{d\phi}{ds} + \sin(2\phi-\zeta) \frac{dP}{ds}] ,$$

(B.2.4a,b)

$$\frac{\partial P}{\partial y} = \frac{1}{\sin 2(\phi-\zeta)} [- \cos \zeta \frac{d\phi}{ds} - \cos (2\phi-\zeta) \frac{dP}{ds}].$$

It follows from equations (B.2.3a,b) and (B.2.4a,b) that $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ are not determinate when $\zeta = \phi$ and $\zeta = \phi + \frac{\pi}{2}$ if

$$dP + d\phi = 0 \quad \text{when } \zeta = \phi$$

and (B.2.5a,b)

$$dP - d\phi = 0 \quad \text{when } \zeta = \phi + \frac{\pi}{2} .$$

Hence the characteristics of the stress field are defined by

$$\frac{dy}{dx} = \tan \phi$$

and (B.2.6a,b)

$$\frac{dy}{dx} = - \cot \phi.$$

Curves defined by equation (B.2.6a) are called α-lines and those defined

by equation (B.2.6b) are called β-lines and the compatibility relationships are given by equation (B.2.5a) and equation (B.2.5b) for the α- and β-lines respectively. For the limiting case of a rigid-plastic solid, the characteristics and compatibility relationships, which are known as the HENCKY equations, are the same as those above.

For any other case, it follows from equation (B.1.6) with use of equations (B.1.7c,d) that

$$\frac{\partial v_x}{\partial x}\,[\sin(2\zeta-2\phi) - \sin 2\gamma] + \sin 2\zeta \sin 2\gamma\,[v_x \frac{\partial\phi}{\partial x} + v_y \frac{\partial\phi}{\partial y}]$$
$$+ 2\cos\zeta\,[\sin 2\gamma + \sin 2\phi]\,\frac{dv_x}{ds} - 2\sin\zeta\,[\sin 2\gamma - \sin 2\phi]\,\frac{dv_y}{ds} = 0. \tag{B.2.7}$$

If $\frac{\partial v_x}{\partial x}$ is determined, then by equations (B.1.7c,d) and (B.1.3d), so are the partial derivatives $\frac{\partial v_x}{\partial y}$, $\frac{\partial v_y}{\partial x}$ and $\frac{\partial v_y}{\partial y}$. In equation (B.2.7) it is seen that when $\zeta = \phi + \gamma$ or $\zeta = \phi - \gamma + \frac{\pi}{2}$, $\frac{\partial v_x}{\partial x}$ (and hence $\frac{\partial v_x}{\partial y}$, $\frac{\partial v_y}{\partial x}$ and $\frac{\partial v_y}{\partial y}$) is not determinate if

$$\sin 2\zeta \sin 2\gamma\,[v_x \frac{\partial\phi}{\partial x} + v_y \frac{\partial\phi}{\partial y}] + \cos\zeta\,[\sin 2\gamma + \sin 2\phi]\,\frac{dv_x}{ds}$$
$$- \sin\zeta\,[\sin 2\gamma - \sin 2\phi]\,\frac{dv_y}{ds} = 0. \tag{B.2.8}$$

Consequently, the characteristics of the velocity field are defined by

$$\frac{dy}{dx} = \tan(\phi + \gamma)$$

and

$$\frac{dy}{dx} = -\cot(\phi - \gamma). \tag{B.2.9a,b}$$

Curves defined by equation (B.2.9a) are called α_1-lines and those defined by equation (B.2.9b) are called β_1-lines. Also, since $\gamma \neq 0$, substitution of equations (B.2.3a,b) into equation (B.2.8) is permissible and the result is

$$\frac{\sin 2\zeta \sin 2\gamma}{\sin 2(\phi-\zeta)} [\{v_x \sin(2\phi-\zeta) - v_y \cos(2\phi-\zeta)\} \frac{d\phi}{ds} + \{v_x \sin\zeta - v_y \cos\zeta\} \frac{dP}{ds}]$$

$$+ \cos\zeta[\sin 2\gamma + \sin\phi] \frac{dv_x}{ds} - \sin\zeta[\sin 2\gamma - \sin 2\phi] \frac{dv_y}{ds} = 0. \qquad (B.2.10)$$

For an α_1-line, $\zeta = \phi + \gamma$ and equation (B.2.10) reduces after simplification to

$$\frac{d}{ds} [v_x \cos(\phi-\gamma) + v_y \sin(\phi-\gamma)]$$

$$+ [v_y \cos(\phi+\gamma) - v_x \sin(\phi+\gamma)] \frac{dP}{ds} = 0. \qquad (B.2.11)$$

On defining

$$U \equiv v_x \cos(\phi-\gamma) + v_y \sin(\phi-\gamma)$$

and (B.2.12a,b)

$$V \equiv v_y \cos(\phi+\gamma) - v_x \sin(\phi+\gamma),$$

equation (B.2.11) becomes

$$\frac{dU}{ds} + V \frac{dP}{ds} = 0$$

or equivalently,

$$dU + \frac{V}{2k}\,dp = 0, \tag{B.2.13}$$

the compatibility relation along an α_1-line. Similarly for a β_1-line for which $\zeta = \phi - \gamma + \frac{\pi}{2}$, equation (B.2.10) reduces to

$$\frac{d}{ds}\left[v_y \cos(\phi+\gamma) - v_x \sin(\phi+\gamma)\right]$$

$$+ \left[v_x \cos(\phi-\gamma) + v_y \sin(\phi-\gamma)\right] \frac{dP}{ds} = 0.$$

With use of equations (B.2.12a,b), this equation is written simply as

$$\frac{dV}{ds} + U\,\frac{dP}{ds} = 0$$

or equivalently

$$dV + \frac{U}{2k}\,dp = 0 , \tag{B.2.14}$$

the compatibility relation along a β_1-line.

Equations (B.2.13) and (B.2.14) are alternate forms of the compatibility relations along an α_1-line and a β_1-line derived in CHAPTER VI, SECTION 6.1. From equations (B.2.12a,b), it is observed that V is the component of the velocity perpendicular to the α_1-line and U is the component of velocity perpendicular to the β_1-line. If, however, the velocity is resolved into two components u and v where u is the component along the α_1-line and v the component along the β_1-line, then

$$U = u \cos 2\gamma$$

and

$$V = v \cos 2\gamma.$$

Use of these relations in equations (B.2.13) and (B.2.14) yield respectively

$$du + v\,\frac{dp}{2k} = 0 \quad \text{along an } \alpha_1\text{-line}$$

and

$$dv + u\,\frac{dp}{2k} = 0 \quad \text{along a } \beta_1\text{-line} \;,$$

thus recovering the form of the compatibility relations along the velocity characteristics derived originally in CHAPTER VI.

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